Exam	Final exam in GRA 6035 Mathematics
Date	November 29th, 2017 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the quadratic form f given by $f(x, y, z, w) = x^2 + y^2 + 9z^2 + 4w^2 + 6yz - 4yw - 12zw$.

(a) **(6p)** Find the symmetric matrix A of the quadratic form f, and compute the rank of A.
(b) **(6p)** Determine the definiteness of the quadratic form f.
(c) **(6p)** Find two vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $\text{span}(\mathbf{v}_1, \mathbf{v}_2)$ is the set of solutions of $A\mathbf{x} = \mathbf{0}$.

Question 2.

Find the general solutions of the following differential equations:

(a) **(6p)** $y' - 2y = e^t$
(b) **(6p)** $3t^2 - y - ty' = 0$

Find all equilibrium states of the following differential equation, and determine their stability. Are any of the equilibrium states globally asymptotically stable?

(c) **(6p)** $y' = 2y(3 - y)$

Question 3.

Let u be the function given by $u(x, y, z) = 1 + x^2 + 5y^2 + 8z^2 + 4xy - 2yz$, and consider the composite function $f(x, y, z) = \ln(u)/u^2$ with $u = u(x, y, z)$.

(a) **(6p)** Find the minimal value of $u = u(x, y, z)$, if it exists.
(b) **(6p)** Compute the first order partial derivatives of $f = f(x, y, z)$.
(c) **(6p)** Determine the maximum and minimum values of f.

Question 4.

We consider the following Kuhn-Tucker problem:

$$\max\ f(x, y) = x^2y^2 \text{ subject to } x^2 + y^2 + x^2y^2 \leq 3$$

(a) **(6p)** Write down all Kuhn-Tucker conditions for this problem.
(b) **(6p)** Find all points $(x, y; \lambda)$ with $x, y \neq 0$ that satisfy the Kuhn-Tucker conditions.
(c) **(6p)** Show that the Kuhn-Tucker problem has a maximum, and find the maximum value.

Question 5.

Extra credit (6p) Find the general solution of following system of differential equations:

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} y \\ z \end{pmatrix}$$