Solutions	Final exam in GRA 6035 Mathematics
Date	December 11th, 2015 at 0900 - 1200

Question 1.

(a) The rank of A is two since it has an echelon form

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The linear system $A\mathbf{x} = \mathbf{0}$ has free variables z, w and the solution is given by the equations $x + w = 0$ and $y + z = 0$, which gives $x = -w$ and $y = -z$, or

$$\mathbf{x} = \begin{pmatrix} -w \\ -z \\ z \\ w \end{pmatrix} = z \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(b) Since A is symmetric, it is diagonalizable. The eigenvalues are given by the characteristic equation $\det(A - \lambda I) = 0$, which becomes

$$\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

This gives the equation

$$-\lambda(-\lambda(\lambda^2 - 1) + 1(\lambda)) - 1(1(\lambda^2 - 1) - 1(-1)) + 1(1(-1) - 1(\lambda^2 - 1)) = 0$$

which gives, after multiplication, that

$$\lambda^2(\lambda^2 - 1) - \lambda^2 - \lambda^2 - \lambda^2 = \lambda^2(\lambda^2 - 1 - 3) = \lambda^2(\lambda^2 - 4) = \lambda^2(\lambda - 2)(\lambda + 2) = 0$$

The eigenvalues are therefore $\lambda = 0$ (with multiplicity two), $\lambda = 2$ and $\lambda = -2$ (both with multiplicity one).

(c) We know that there is an invertible matrix P such that $P^{-1}AP = D$ is diagonal, since A is diagonalizable. Therefore $D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$, so $B = A^2$ is also diagonalizable. We have that

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \Rightarrow \quad D^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

The eigenvaluess of $B = A^2$ are therefore $\lambda = 0$ (with multiplicity two) and $\lambda = 4$ (with multiplicity two). Alternatively, one could answer this question by computing

$$B = A^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Since B is symmetric, it is diagonalizable. We could find the eigenvalues of B by solving the characteristic equation $\det(B - \lambda I) = 0$, which gives $\lambda^2(\lambda - 4)^2 = 0$, or $\lambda = 0$ (with multiplicity two) and $\lambda = 4$ (with multiplicity two).

(d) We know that $\mathbf{x}_{t+1} = T\mathbf{x}_t$ is a regular Markov chain, since all entries in T are positive, so the equilibrium state $\mathbf{x}$ is the unique eigenvalue of T with eigenvalue $\lambda = 1$ that is a state vector. We compute the eigenvectors with eigenvalue $\lambda = 1$:

$$\begin{pmatrix} -0.45 & 0.10 & 0.15 \\ 0.10 & -0.20 & 0.05 \\ 0.35 & 0.10 & -0.20 \end{pmatrix} \to \begin{pmatrix} -45 & 10 & 15 \\ 10 & -20 & 5 \\ 35 & 10 & -20 \end{pmatrix} \to \begin{pmatrix} 10 & -20 & 5 \\ -45 & 10 & 15 \\ 35 & 10 & -20 \end{pmatrix}$$

$$\to \begin{pmatrix} 10 & -20 & 5 \\ 0 & -80 & 37.5 \\ 0 & 80 & -37.5 \end{pmatrix} \to \begin{pmatrix} 10 & -20 & 5 \\ 0 & -160 & 75 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that z is free, that $y = 75z/160$ and that $10x = 20(75z/160) - 5z = 35z/8$, so $x = 35z/80 = 70z/160$. The unique eigenvector that is a state vector is given by

$$x + y + z = 1 \quad \Rightarrow \quad \frac{70z + 75z + 160z}{160} = \frac{305z}{160} = 1$$

which gives $z = 160/305$. This gives equilibrium state $\mathbf{x}$ with

$$x = \frac{70}{305} = \frac{14}{61} \approx 0.23, \quad y = \frac{75}{305} = \frac{15}{61} \approx 0.25, \quad z = \frac{160}{305} = \frac{32}{61} \approx 0.52$$

Question 2.

(a) The differential equation $y'' - 4y' - 12y = 15e^t$ is second order linear, and has general solution $y = y_h + y_p$. The homogeneous solution is

$$y_h = C_1e^{6t} + C_2e^{-2t}$$

since the characteristic equation $r^2 - 4r - 12 = 0$ has solutions $r = 6$ and $r = -2$. To find the particular solution, we guess a solution of the form $y = Ae^t$, since $f(t) = 15e^t = f'(t) = f''(t)$. We compute $y' = y'' = Ae^t$, which gives

$$Ae^t(1 - 4 - 12) = 15e^t \quad \Rightarrow \quad -15Ae^t = 15e^t$$

We see that $A = -1$ is a solution, so $y_P = -e^t$ and the general solution is

$$y = C_1e^{6t} + C_2e^{-2t} - e^t$$

(b) The differential equation $y' = 3\sqrt{t} \cdot e^{-2y}$ is separable, and it can be written in the form

$$e^{2y}y' = 3t^{1/2} \quad \Leftrightarrow \quad \int e^{2y}\,dy = \int 3t^{1/2}\,dt$$

Integration gives $e^{2y}/2 = 2 \cdot t^{3/2} + \mathcal{C}$, and therefore that

$$e^{2y} = 4t^{3/2} + 2\mathcal{C} = 4t\sqrt{t} + 2\mathcal{C} \quad \text{or} \quad y = \frac{1}{2}\ln\left(4t\sqrt{t} + 2\mathcal{C}\right)$$

(c) The differential equation $4yt + 4t^3 + 2t + (2y - 1 + 2t^2)y' = 0$ can be written in the form $p + qy' = 0$ with

$$p = 4yt + 4t^3 + 2t, \quad q = 2y - 1 + 2t^2$$

We attempt to find a function $h = h(y,t)$ such that $h'_t = p$ and $h'_y = q$. From the first equation, we see that $h = 2yt^2 + t^4 + t^2 + \phi(y)$, since $(2yt^2 + t^4 + t^2 + \phi(y))'_t = 4yt + 4t^3 + 2t = p$. Using this expression for h, the second condition becomes

$$h'_y = 2t^2 + \phi'(y) = 2y - 1 + 2t^2$$

which is satisfied if $\phi'(y) = 2y - 1$, and one solution is $\phi(y) = y^2 - y$. This implies that differential equation $p + qy' = 0$ is exact and that $h = 2yt^2 + t^4 + t^2 + y^2 - y$ satisfies $h'_t = p$ and $h'_y = q$. The solution of the differential equation is therefore

$$2yt^2 + t^4 + t^2 + y^2 - y = \mathcal{C}$$

The inital condition $y(1) = 0$ gives that $2 = \mathcal{C}$, so we have that

$$2yt^2 + t^4 + t^2 + y^2 - y = 2 \quad \Rightarrow \quad y^2 + (2t^2 - 1)y + (t^4 + t^2 - 2) = 0$$

To find an explicit solution, we solve for y using the abc-formula:

$$y = \frac{-(2t^2-1) \pm \sqrt{(2t^2-1)^2 - 4(t^4+t^2-2)}}{2} = \frac{1-2t^2 \pm \sqrt{9-8t^2}}{2}$$

Using the inital condition $y(1) = 0$ again, we see that the particular solution of the differential equation is

$$y = \frac{1-2t^2+\sqrt{9-8t^2}}{2}$$

Question 3.

(a) To determine the definiteness of the quadratic form u, we may use the symmetric matrix A of u, given by

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

or alternatively the Hessian matrix of u, which is $2A$. We find that $D_1 = 2$, $D_2 = 5$ and $D_3 = 2$. Therefore, u is positiv definite. This means that $u(x,y,z) \geq 0$ for all (x,y,z), with $u > 0$ when $(x,y,z) \neq (0,0,0)$. So $u+1 \geq 1$, and $f(x,y,z) = \ln(u+1)$ is defined for all points (x,y,z) in $\mathbb{R}^3$ since the natural logarithm is defined for all positive numbers.

(b) The partial derivatives of $f(x,y,z) = \ln(u+1)$ are given by

$$f'_x = \frac{4x+2y-2z}{u+1}, \quad f'_y = \frac{2x+6y}{u+1}, \quad f'_z = \frac{-2x+2z}{u+1}$$

Since $u+1 \geq 1$, the stationary points are the solutions to the linear equations

$$4x+2y-2z = 0, \quad 2x+6y = 0, \quad -2x+2z = 0$$

We may observe that this is the linear system $(2A)\mathbf{x} = \mathbf{0}$, and since $D_3 = |A| \neq 0$, this gives $\mathbf{x} = \mathbf{0}$. Or we may solve the equations: The last two give $z = x$ and $y = -x/3$, and when we substitute this in the first equation, we get

$$4x + 2(-x/3) - 2x = 0 \quad \Rightarrow \quad 4x/3 = 0 \quad \Rightarrow \quad x = 0$$

This means that $x = y = z = 0$, so there is only one stationary point $(x,y,z) = (0,0,0)$.

(c) The stationary point $(x,y,z) = (0,0,0)$ has function value $f(0,0,0) = \ln(1) = 0$, and when $(x,y,z) \neq (0,0,0)$ we have that $u(x,y,z) > 0$ and hence that $f(x,y,z) = \ln(u+1) > \ln(1) = 0$. Therefore $(0,0,0)$ is the minimizer of f, with minimal value $f(0,0,0) = 0$. To check if f is convex, we start by computing $D_1 = f''_{xx}$:

$$D_1 = f''_{xx} = \left(\frac{4x+2y-2z}{u+1}\right)'_x = \frac{4(2x^2+2xy+3y^2-2xz+z^2+1) - (4x+2y-2z)^2}{(u+1)^2}$$

Let for example $x = 1$, $y = z = 0$. Then D_1 has value

$$D_1 = \frac{4(2+1) - 4^2}{(2+1)^2} = -\frac{4}{9} < 0$$

It follows that the requirement $D_1 \geq 0$ for all (x,y,z) is not satisfied. Therfore, f is not convex.

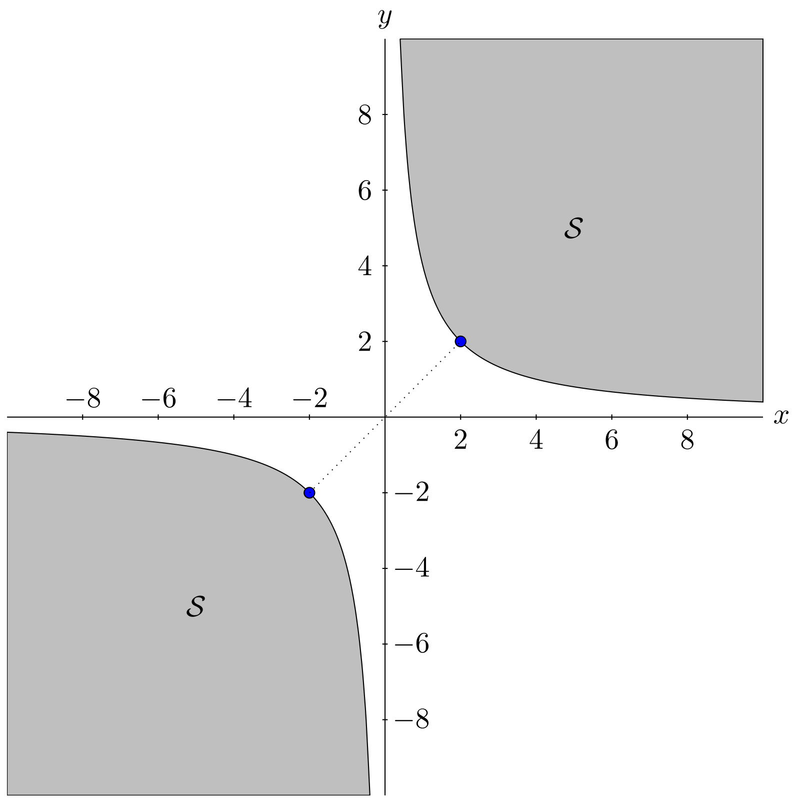
Question 4.

(a) The boundary of the set of S of admissible points is given by the equation $xy = 4$, and it is therefore the graph of $y = 4/x$, an hyperbola. This hyperbola, and the region S of admissible points, is shown in the figure below. When $x > 0$, $xy > 4$ gives $y > 4/x$, so the region with $xy > 4$ lies above the hyperbola. When $x < 0$, $xy > 4$ gives $y < 4/x$, so the region with $xy > 4$ lies below the hyperbola. Even though the drawing only shows points with $-10 \leq x, y, \leq 10$, we see that the region S is not bounded. In fact, the point $(x,y) = (a,a)$ is in S when $a \to \infty$

since $xy = a^2 \geq 4$ when $a \geq 2$. The points (x, y) in S that minimizes the distance $d(x, y)$ to the origin are the points that minimizes $f(x, y)$, since

$$f(x,y) = x^2 + y^2 = \left(\sqrt{x^2 + y^2}\right)^2 = d(x,y)^2$$

It is clear from the figure that there are two points (x, y) that minimizes this distance. In fact, it seems from the drawing that those points are $(2, 2)$ and $(-2, -2)$.



(b) We write the Kuhn-Tucker problem in standard form as

$$\max\ -f(x,y) = -x^2 - y^2 \text{ subject to } -xy \leq -4$$

It has Lagrangian $\mathcal{L} = -x^2 - y^2 - \lambda(-xy) = -x^2 - y^2 + \lambda xy$. The first order conditions (FOC) are

$$\mathcal{L}'_x = -2x + \lambda y = 0$$
$$\mathcal{L}'_y = -2y + \lambda x = 0$$

the constraint (C) is given by $xy \geq 4$, and the complementary slackness conditions (CSC) are given by

$$\lambda \geq 0 \quad \text{and } \lambda(xy - 4) = 0$$

The FOC's give that $x = \lambda y/2$ and $-2y + \lambda x = -2y + \lambda(\lambda y/2) = 0$, or $y/2 \cdot (-4 + \lambda^2) = 0$. This gives $y = 0$ or $\lambda = \pm 2$. If $y = 0$, then $x = 0$ by the FOC's, and $xy = 0$ does not satisfy the constraint. If $\lambda = \pm 2$, then we must have $\lambda = 2$ and $xy = 4$ by the CSC's. In this case, the FOC's give $x = y$, which means that $xy = x^2 = 4$ and that $x = \pm 2$. We find two solutions of the Kuhn-Tucker conditions:

$$(x, y; \lambda) = (2, 2; 2), \quad (x, y; \lambda) = (-2, -2; 2)$$

Both points give $f(x, y) = 2^2 + 2^2 = 8$. To show that these points are actually the minimizers of f, we can apply SOC: With $\lambda = 2$, the Lagrangian $\mathcal{L}(x, y; 2) = -x^2 - y^2 + 2xy$. This is clearly a concave function since it has Hessian

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

with $D_1 = -2$ and $D_2 = 0$, and $\Delta_1 = -2, -2$ and $\Delta_2 = 0$. Therefore $(x, y) = (2, 2), (-2, -2)$ are maximizers for $-f$ and minimizers of f. Alternatively, we could prove this by invoking the geometric argument from a) that f has a minimum, and show that there are no admissible points where NDCQ fails since

$$\text{rk}\begin{pmatrix} y & x \end{pmatrix} < 1$$

means that $x = y = 0$, which is not an admissible point. Therefore, the minimum is obtained at the points $(2, 2)$ and $(-2, -2)$, the only points that that satisfy the Kuhn-Tucker conditions FOC+C+CSC.

(c) We write the Kuhn-Tucker problem in standard form as

$$\max\ -f(x, y, z, w) = -(x^2 + y^2 + z^2 + w^2) \text{ subject to } \begin{cases} -xz \le -9 \\ -yw \le -25 \end{cases}$$

Therefore, the Lagrangian of the problem is given by

$$\begin{aligned} \mathcal{L} &= -(x^2 + y^2 + z^2 + w^2) - \lambda_1(-xz) - \lambda_2(-yw) \\ &= -x^2 - y^2 - z^2 - w^2 + \lambda_1 xz + \lambda_2 yw \end{aligned}$$

The first order conditions (FOC) are

$$\begin{aligned} \mathcal{L}'_x &= -2x + \lambda_1 z = 0 \\ \mathcal{L}'_y &= -2y + \lambda_2 w = 0 \\ \mathcal{L}'_z &= -2z + \lambda_1 x = 0 \\ \mathcal{L}'_w &= -2w + \lambda_2 y = 0 \end{aligned}$$

the constraints (C) is given by $xz \ge 9$ and $yw \ge 25$, and the complementary slackness conditions (CSC) are given by

$$\begin{aligned} \lambda_1 \ge 0 \quad &\text{and } \lambda_1(xz - 9) = 0 \\ \lambda_2 \ge 0 \quad &\text{and } \lambda_2(yw - 25) = 0 \end{aligned}$$

From the first two FOC's we see that $x = \lambda_1 z/2$ and $y = \lambda_2 w/2$. When we substitute this into the last two FOC's, we get

$$-2z + \lambda_1(\lambda_1 z/2) = -\frac{z}{2}(4 - \lambda_1^2) = 0$$

and

$$-2w + \lambda_2(\lambda_2 w/2) = -\frac{w}{2}(4 - \lambda_2^2) = 0$$

From the first equation, we get $z = 0$ or $\lambda_1 = \pm 2$. If $z = 0$, then $xz = 0$, and the constraint $xz \ge 9$ is not satisfied. Therefore, we get $\lambda_1 = 2$ and $xz = 9$ from the first CSC's. The FOC's then give $x = z$, so that $x^2 = 9$ and $x = z = \pm 3$. From the second equation, we get $w = 0$ or $\lambda_2 = \pm 2$. If $w = 0$, then $yw = 0$, and the constraint $yw \ge 25$ is not satisfied. Therefore, we get $\lambda_2 = 2$ and $yw = 25$ from the last CSC's. The FOC's then give $y = w$, so that $y^2 = 25$ and $y = w = \pm 5$. From all of this, we get the following solutions to the Kuhn-Tucker conditions FOC+C+CSC:

$$\begin{aligned} (x, y, z, w; \lambda_1, \lambda_2) = &(3, 5, 3, 5; 2, 2), (3, -5, 3, -5; 2, 2), \\ &(-3, 5, -3, 5; 2, 2), (-3, -5, -3, -5; 2, 2) \end{aligned}$$

At all four points we have $f(x, y, z, w) = 3^2 + 5^2 + 5^2 + 3^2 = 68$. To show the these four points minimizes f, we apply the SOC: We consider the Lagrangian

$$\mathcal{L}(x, y, z, w; 2, 2) = -x^2 - y^2 - z^2 - w^2 + 2xz + 2yw$$

It has Hessian

$$H = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

It has leading principal minors $D_1 = -2$, $D_2 = 4$, $D_3 = 0$ and $D_4 = 0$. To compute all principal minors, we notice that $rkH = 2$, so that $\Delta_3 = 0$ and $\Delta_4 = 0$ for all principal minors of order 3 or 4. Furthermore, we have $\Delta_1 = -2, -2, -2, -2 \leq 0$ and $\Delta_2 = 4, 0, 4, 4, 0, 4 \geq 0$. It follows that H is negative semidefinite, and therefore that the four points above maximizes $-f$, and minimizes f. The minimium value is therefore $f = 68$.