Solutions:	**GRA 60353** Mathematics	
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Answer sheets:	Squares	
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Ordinary exam		Responsible department: Economics

Question 1.

(a) The partial derivatives of $f(x,y,z,w) = xw - yz$ are given by

$$f'_x = w, \quad f'_y = -z, \quad f'_z = -y, \quad f'_w = x$$

and its Hessian matrix is given by

$$H(f)(x,y,z,w) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(b) The stationary points of f are given by

$$f'_x = w = 0, \quad f'_y = -z = 0, \quad f'_z = -y = 0, \quad f'_w = x = 0$$

and there is a unique solution $(x,y,z,w) = (0,0,0,0)$. The Hessian matrix at this point is the symmetric matrix

$$H(f)(0,0,0,0) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We have that $D_1 = D_2 = D_3 = 0$ and $D_4 = \det(A) = (-1)(-1)\cdot 1 \cdot 1 = 1$, and we must check the remaining principal minors to determine the definiteness of the matrix. All principal minors of order one (the elements on the diagonal) are zero, but there is a second order principal minor that is negative; the principal minor obtained by keeping row $2,3$ and column $2,3$ in A is $\Delta_2 = -1$. This means that the Hessian matrix at $(0,0,0,0)$ is indefinite, and therefore the stationary point $(0,0,0,0)$ is a saddle point.

(c) If there were a global max for f, it would also be a local max. But there is no local max, since the only stationary point for f is a saddle point. Therefore, f has no global max.
Altenative: Consider points where $x = w = a$ and $y = z = 0$ for some number a. Then the value of f is

$$f(x,y,z,w) = f(a,0,0,a) = a^2$$

and $a^2 \to \infty$ as $a \to \infty$. This implies that choosing points along a certain trajectory, the values of f goes toward ∞ and therefore f has no global maximum value.

Question 2.

(a) The matrices $A+I$ and $A-I$ are given by

$$A+I = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad A-I = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

In both cases, Gaussian elimination gives an echelon form with zeros in the last two rows. Therefore, both matrices have two pivot positions and $\text{rk}(A+I) = \text{rk}(A-I) = 2$.

(b) The matrix A is symmetric, therefore it is diagonalizable.
Alternative 1: Since $\text{rk}(A+I) = \text{rk}(A-I) = 2$, we see that $\det(A-\lambda I) = 0$ when $\lambda = \pm 1$, and this implies that $\lambda = -1$ and $\lambda = 1$ are eigenvalues of A. Moreover, there are two linearly independent eigenvectors for each of these eigenvectors since the number of free variables is given by $n - \text{rk}(A - \lambda I) = 4 - 2 = 2$ in each case by (a). Therefore there are enough eigenvalues and enough eigenvectors, and A is diagonalizable.
Alternative 2: Find all eigenvalues using the characteristic equation. Its left side is given by

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & -1 & 0 \\ 0 & -1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 & 1 \\ -\lambda & -1 & 0 \\ -1 & -\lambda & 0 \end{vmatrix}$$

Hence the characteristic equation is $\lambda^2(\lambda^2-1) - 1(\lambda^2-1) = 0$, or $(\lambda^2-1)(\lambda^2-1) = 0$, and the eigenvalues are $\lambda = 1$ and $\lambda = -1$, both with multiplicity two. By the same argument as in Alternative 1, there are $4-2 = 2$ free variable in $(A-\lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda = 1$ and $\lambda = -1$. Hence there are enough eigenvalues and eigenvectors, and A is diagonalizable.

(c) We compute the eigenvectors for $\lambda = -1$ by solving the linear system $(A+I)\mathbf{x} = \mathbf{0}$. Since $\text{rk}(A+I) = 2$, there are $4-2 = 2$ free variables, and since we got pivot positions at $(1,1)$ and $(2,2)$ in (a), we may take x_3 and x_4 as free variables and solve for x_1 and x_2 in the first two equations to get $x_1 = -x_4$ and $x_2 = x_3$. The eigenvectors for $\lambda = -1$ are therefore given by

$$\mathbf{x} = \begin{pmatrix} -x_4 \\ x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Question 3.

(a) The difference equation $y_{t+2} - 11y_{t+1} + 28y_t = 36t + 18$ is second order linear, and it has solution $y_t = y_t^h + y_t^p$. The homogeneous equation $y_{t+2} - 11y_{t+1} + 28y_t = 0$ has characteristic equation $r^2 - 11r + 28 = 0$, and distinct roots $r = 4$ and $r = 7$. Therefore $y_t^h = C_1 \cdot 4^t + C_2 \cdot 7^t$. To find a particular solution y_t^p, we consider the right hand side $f_t = 36t + 18$ and the shifted expressions $f_{t+1} = 36t + 54$ and $f_{t+2} = 36t + 90$. We guess that there is a solution of the form $y_t = At + B$. Inserting this guess in the difference equation, we obtain

$$(At + B + 2A) - 11(At + B + A) + 28(At + B) = 36t + 18$$

or $(18A)t + (-9A + 18B) = 36t + 18$. We see that $A = 2$ and $B = 2$ is a solution, so $y_t^p = 2t + 2$ and the general solution is

$$y_t = y_t^h + y_t^p = C_1 \cdot 4^t + C_2 \cdot 7^t + 2t + 2$$

(b) The differential equation $y' = 4y + te^t$ is first order linear, and can be written in standard form as $y' - 4y = te^t$. It has solution $y = y_h + y_p$, and the homogeneous solutions $y_h = Ce^{4t}$ since the characteristic equation $r - 4 = 0$ of the homogeneous equation has root $r = 4$. To find a particular solution, we consider the right hand side $f(t) = te^t$ and its derivatives $f' = (t+1)e^t$ and $f'' = (t+2)e^t$. We guess that there is a solution of the form $y = (At+B)e^t$. Inserting this guess in the differential equation, we obtain

$$(At + B + A)e^t - 4(At + B)e^t = te^t$$

or $(-3At + A - 3B)e^t = te^t$. We see that there is a solution with $-3A = 1$ and $A - 3B = 0$, or $A = -1/3$ and $B = -1/9$. This means that $y_p = (-3t-1)e^t/9$ is a particular solution, and that the general solution is

$$y = y_h + y_p = Ce^{4t} - \frac{3t+1}{9}\,e^t$$

Alternative: It can also be solved using integrating factor e^{-4t}. We get the general solution

$$y = \frac{1}{e^{-4t}} \int te^{-3t}\,dt = e^{4t}\left(-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + \mathcal{C}\right) = \mathcal{C}e^{4t} - \frac{3t+1}{9}\,e^t$$

(c) The differential equation can be written in the form $py' + q = 0$ with

$$p = \frac{y}{y^2+t^2+3}, \qquad q = \frac{t}{y^2+t^2+3}$$

We attempt to find an expression $h = h(y,t)$ such that $h'_y = p$ and $h'_t = q$. From the first equation, we see that $h = \frac{1}{2}\ln(y^2 + t^2 + 3) + \phi(t)$ is a solution, and using this expression for h we get

$$h'_t = \frac{t}{y^2+t^2+3} + \phi'(t), \quad q = \frac{t}{y^2+t^2+3}$$

This implies that the second equation $h'_t = q$ is satisfied if $\phi'(t) = 0$, and we may choose $\phi(t) = 0$ and $h = \ln(y^2+t^2+3)/2$. Hence the differential equation is exact, and its solution is given by

$$\ln(y^2+t^2+3)/2 = C \quad \Leftrightarrow \quad y^2 + t^2 = e^{2C} - 3 = K$$

In other words, $y = \pm\sqrt{K - t^2}$. The initial condition $y(1) = 2$ gives $2 = +\sqrt{K-1}$ or $K = 5$, and the particular solution is

$$y = \sqrt{5 - t^2}$$

Question 4.

(a) The Kuhn-Tucker problem is already in standard form, so we form the Lagrangian

$$\mathcal{L} = xw - yz - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$$

The first order conditions (FOC) are

$$\begin{aligned}
\mathcal{L}'_x &= w - 2\lambda_1 x = 0\\
\mathcal{L}'_y &= -z - 2\lambda_1 y = 0\\
\mathcal{L}'_z &= -y - 8\lambda_2 z = 0\\
\mathcal{L}'_w &= x - 18\lambda_2 w = 0
\end{aligned}$$

the constraints (C) are given by $x^2 + y^2 \leq 1$ and $4z^2 + 9w^2 \leq 36$, and the complementary slackness conditions (CSC) are given by

$$\begin{aligned}
\lambda_1 \geq 0 \quad &\text{and } \lambda_1(x^2+y^2-1) = 0\\
\lambda_2 \geq 0 \quad &\text{and } \lambda_2(4z^2+9w^2-36) = 0
\end{aligned}$$

When $(x,y,z,w) = (0,1,-3,0)$, we see that the the first and last of the FOC's are satisfied, and the remaining FOC's are satisfied if $\lambda_1 = 3/2$ and $\lambda_2 = 1/24$. The constraints are satisfied (and binding), and this implies that the CSC are satisfied as well since $\lambda_1, \lambda_2 \geq 0$.

(b) It follows from the SOC that $(x, y, z, w) = (0, 1, -3, 0)$ solves the max problem if the function $\mathcal{L}(x, y, z, w; 3/2, 1/24)$ is concave in (x, y, z, w). We prove that this is the case: The function is given by

$$\mathcal{L}(x, y, z, w; \lambda_1^*, \lambda_2^*) = xw - yz - 3/2(x^2 + y^2) - 1/24(4z^2 + 9w^2)$$

Its Hessian matrix is given by

$$H = \begin{pmatrix} -3 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -1 & -1/3 & 0 \\ 1 & 0 & 0 & -3/4 \end{pmatrix}$$

The leading principal minors are $D_1 = -3$, $D_2 = 9$, $D_3 = 0$ and $D_4 = 0$, so we have to compute all principal minor to be sure that the function is concave. The four principal minors of order one are

$$\Delta_1 = -3, -3, -1/3, -3/4 \leq 0$$

the six principal minors of order two are

$$\Delta_2 = 9, 1, 5/4, 0, 9/4, 1/4 \geq 0$$

and the four principal minors of order three are

$$\Delta_3 = 0, 0, -15/4, -5/12 \leq 0$$

and the only principal minor of order four is $D_4 = 0 \geq 0$. It follows that the Hessian is negative semidefinite, and therefore that $\mathcal{L}(x, y, z, w; \lambda_1^*, \lambda_2^*)$ is concave in (x, y, z, w). Hence $(x, y, z, w) = (0, 1, -3, 0)$ is max, with max value $f(0, 1, -3, 0) = 3$.

(c) We consider the Kuhn-Tucker problem with parameter c given by

$$\max\ f(x, y, z, w) = xw - yz \text{ subject to } \begin{cases} x^2 + y^2 \leq 1 \\ cz^2 + 9w^2 \leq 36 \end{cases}$$

which we have solved for $c = 4$. It has Lagrangian

$$\mathcal{L} = xw - yz - \lambda_1(x^2 + y^2) - \lambda_2(cz^2 + 9w^2)$$

and therefore $\mathcal{L}'_c = -\lambda_2 z^2$. By the Envelope Theorem, the maximum value changes with approximately

$$\Delta c \cdot \mathcal{L}'_c(x^*, y^*, z^*, w^*; \lambda_1^*, \lambda_2^*) = (4.2 - 4) \cdot (-1/24 \cdot (-3)^2) = -0.075$$

when c changes from $c = 4$ to $c = 4.2$. The new maximum value is therefore approximately equal to $3 - 0.075 = 2.925$. (And it is exactly equal to $\sqrt{36/4.2} \cong 2.9277$).

(d) To solve the Kuhn-Tucker conditions (FOC+C+CSC) in a), we start with the FOC's: From the first and last of them, we see that

$$w = 2\lambda_1 x, \quad x = 18\lambda_2 w \quad \Rightarrow \quad w = 36\lambda_1\lambda_2 w$$

This implies that $w(1-36\lambda_1\lambda_2) = 0$, so either $w = 0$ (which implies that $x = 0$) or $\lambda_1\lambda_2 = 1/36$. From the second and third FOC's, we get in a similar way that

$$z = -2\lambda_1 y, \quad y = -8\lambda_2 z \quad \Rightarrow \quad z = 16\lambda_1\lambda_2 z$$

This implies that $z(1-16\lambda_1\lambda_2) = 0$, so either $z = 0$ (which implies that $y = 0$) or $\lambda_1\lambda_2 = 1/16$.
Case A: $x = y = z = w = 0$ In this case, all FOC's are satisfied, both constraints are satisfied and not binding, so $\lambda_1 = \lambda_2 = 0$ by the CSC's. We obtain one solution

$$(x, y, z, w; \lambda_1, \lambda_2) = (0, 0, 0, 0; 0, 0), \quad f = 0$$

Case B: $x = w = 0, z \neq 0$ In this case, $\lambda_1\lambda_2 = 1/16$ since $z \neq 0$. This implies that $\lambda_1, \lambda_2 > 0$ by the CSC's, and both constraints must be binding, so $x^2 + y^2 = 1$ and $4z^2 + 9w^2 = 36$. Since $x = 0$, we must have $y = \pm 1$, and since $w = 0$ we must have $z = \pm 3$. From the FOC $z = -2\lambda_1 y$, we see that y and z must have oposite signs, and that $\lambda_1 = 3/2$ and therefore that $\lambda_2 = 1/24$. We obtain two solutions

$$(x, y, z, w; \lambda_1, \lambda_2) = (0, \pm 1, \mp 3, 0; 3/2, 1/24), \quad f = 3$$

Case C: $y = z = 0, w \neq 0$ In this case, $\lambda_1\lambda_2 = 1/36$ since $w \neq 0$. This implies that $\lambda_1, \lambda_2 > 0$ by the CSC's, and both constraints must be binding, so $x^2 + y^2 = 1$ and $4z^2 + 9w^2 = 36$. Since $y = 0$, we must have $x = \pm 1$, and since $z = 0$ we must have $w = \pm 2$. From the FOC $w = 2\lambda_1 x$, we see that w and x must have the same sign, and that $\lambda_1 = 1$ and therefore that $\lambda_2 = 1/36$. We obtain two solutions

$$(x, y, z, w; \lambda_1, \lambda_2) = (\pm 1, 0, 0, \pm 2; 1, 1/36), \quad f = 2$$