BI NORWEGIAN BUSINESS SCHOOL

Department of Economics

Solutions:	**GRA 60353 Mathematics**
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Question 1.

(a) We compute the partial derivatives $f'_x = 1 - 1/d$, $f'_y = 1 - 2/d$ and $f'_z = 1 - 3/d$, where we write $d = x + 2y + 3z$. The stationary points are given by the equations

$$1 - 1/d = 0, \quad 1 - 2/d = 0, \quad 1 - 3/d = 0$$

and this set of equations have no solutions (the first equation gives $d = 1$, and this does not fit in the other equations). There are therefore **no stationary points**.

(b) We compute the second order partial derivatives of f and form the Hessian matrix

$$f'' = \begin{pmatrix} 1/d^2 \cdot 1 & 1/d^2 \cdot 2 & 1/d^2 \cdot 3 \\ 2/d^2 \cdot 1 & 2/d^2 \cdot 2 & 2/d^2 \cdot 3 \\ 3/d^2 \cdot 1 & 3/d^2 \cdot 2 & 3/d^2 \cdot 3 \end{pmatrix} = \frac{1}{d^2} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

We see that the matrix has rank one, so all second and third order principal minors are 0. The first order principal minors are $1/d^2, 4/d^2, 9/d^2 > 0$. This implies that f is **convex but not concave**.

Question 2.

(a) To find the eigenvalues of A, we solve the characteristic equation $\det(A - \lambda I) = 0$, and this gives

$$\begin{vmatrix} 3-\lambda & 4 & 5 \\ 0 & 2-\lambda & 0 \\ 1 & 3 & 7-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 10\lambda + 16) = 0 \quad \Rightarrow \quad \lambda = 2, \lambda = 2, \lambda = 8$$

This means that the eigenvalues of A are $\lambda = \mathbf{2}, \mathbf{2}, \mathbf{8}$ ($\lambda = 2$ has multiplicity two) and the determinant is $\det(A) = 2 \cdot 2 \cdot 8 = \mathbf{32}$. Since $\det(A) \neq 0$, we have $\operatorname{rk} A = \mathbf{3}$.

(b) The eigenvalues for $\lambda = 2$ are given by $(A - 2I)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} 1 & 4 & 5 \\ 0 & 0 & 0 \\ 1 & 3 & 5 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{t} \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

where t is a free variable. Similarly, the eigenvalues for $\lambda = 8$ are given by $(A - 8I)\mathbf{x} = \mathbf{0}$, or

$$\begin{pmatrix} -5 & 4 & 5 \\ 0 & -6 & 0 \\ 1 & 3 & -1 \end{pmatrix} \mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \mathbf{x} = \mathbf{t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

where t is a free variable. Since $\lambda = 2$ has multiplicity 2 and only has one linearly independent eigenvector (one free variable), A is **not diagonalizable**.

(c) If there is a common eigenvector for A and B, one of the eigenvectors for A must also be an eigenvector for B. In this case, either

$$\mathbf{x}_1 = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \text{ or } \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

must be an eigenvector for B, since any (non-zero) scalar multiple of an eigenvector is an eigenvector. We check if this is the case and start with $\mathbf{x}_1$:

$$B\mathbf{x}_1 = \begin{pmatrix} 0 & 1 & 5 \\ 1 & 3 & 5 \\ 1 & 7 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} = -1 \cdot \mathbf{x}_1$$

Therefore, it follows that $\mathbf{x}_1$ **is a common eigenvector for** A **and** B. (In fact, any vector of the form $t\mathbf{x}_1$ is a common eigenvector. On the other hand, if we do the same computation for $\mathbf{x}_2$, we see that it is not an eigenvector of B, and this means that the vectors $t\mathbf{x}_2$ are not common eigenvectors for A and B.) Finally if $\mathbf{x}$ is an eigenvector for A with eigenvalue λ and an eigenvector for B with eigenvalue λ', then

$$(AB)\mathbf{x} = A(B\mathbf{x}) = A(\lambda'\mathbf{x}) = \lambda'(A\mathbf{x}) = \lambda'(\lambda\mathbf{x}) = (\lambda\lambda')\mathbf{x}$$

This means that $\mathbf{x}$ **is also an eigenvector for** AB (with eigenvalue $\lambda\lambda'$).

Question 3.

(a) We re-write the differential equation as

$$(x+1)t\dot{x} + (t+1)x = 0 \quad \Rightarrow \quad (x+1)\dot{x} = -\frac{(t+1)x}{t} \quad \Rightarrow \quad \frac{x+1}{x}\dot{x} = -\frac{t+1}{t}$$

This differential equation is separated, so the original difference equation is **separable**. We integrate on both sides:

$$\int (1 + \frac{1}{x})\,\mathrm{d}x = -\int (1 + \frac{1}{t})\,\mathrm{d}t \quad \Rightarrow \quad x + \ln(|x|) = -(t + \ln(|t|)) + \mathcal{C}$$

The initial condition $x(1) = 1$ gives $1 + \ln 1 = -1 - \ln 1 + \mathcal{C}$, or $\mathcal{C} = 2$. This solution can therefore be described implicitly by the equation

$$\mathbf{x} + \mathbf{t} + \ln|\mathbf{x}| + \ln|\mathbf{t}| = \mathbf{2}$$

It is not necessary (or possible) to solve this equation for x.

(b) We try to multiply the differential equation by e^{x+t} and get the new differential equation

$$(x+1)te^{x+t}\dot{x} + (t+1)xe^{x+t} = P(x,t)\dot{x} + Q(x,t) = 0$$

with $P(x,t) = (x+1)te^{x+t}$ and $Q(x,t) = (t+1)xe^{x+t}$. We have

$$P'_t = (x+1)e^{x+t} + t(x+1)e^{x+t} = (t+1)(x+1)e^{x+t}$$

and

$$Q'_x = (t+1)e^{x+t} + x(t+1)e^{x+t} = (t+1)(x+1)e^{x+t}$$

We see that $P'_t = Q'_x$, and it follows that the new differential equation is **exact**. To solve it, we find a function $h(x,t)$ such that $h'_x = P(x,t)$ and $h'_t = Q(x,t)$. The first equation gives

$$h'_x = P(x,t) = (x+1)te^{x+t} \quad \Rightarrow \quad h = \int (x+1)te^{x+t}\,\mathrm{d}x = te^t \int (x+1)e^x\,\mathrm{d}x$$

Using integration by parts, we find

$$\int (x+1)e^x\,\mathrm{d}x = (x+1)e^x - \int 1 \cdot e^x\,\mathrm{d}x = (x+1)e^x - e^x + \mathcal{C} = xe^x + \mathcal{C}$$

This implies that

$$h = te^t \int (x+1)e^x\,\mathrm{d}x = te^t xe^x + \mathcal{C}(t) = txe^{x+t} + \mathcal{C}(t)$$

where $\mathcal{C}(t)$ is a function of t (or a constant considered as a function in x). The second equation is $h'_t = Q(x,t)$, and we use the expression above for h:

$$h'_t = Q(x,t) \quad \Rightarrow \quad xe^{x+t} + txe^{x+t} + \mathcal{C}'(t) = (t+1)xe^{x+t} + \mathcal{C}'(t) = (t+1)xe^{x+t}$$

We see that this condition holds if and only if $\mathcal{C}'(t) = 0$, or if $\mathcal{C} = C_1$ is a constant. In conclusion, we may choose $h = txe^{x+t} + C_1$, and the general solution of the exact differential equation is $h = C_2$, where C_2 is another constant. This gives

$$txe^{x+t} = B$$

where $B = C_2 - C_1$ is a new constant. The initial condition is $x(1) = 1$, and this gives $1 \cdot e^2 = B$, or $B = e^2$. The solution can therefore be written in implicit form as

$$\mathbf{txe^{x+t} = e^2}$$

It is not necessary (or possible) to solve this equation for x. (If we first take absolute values on both sides of the equation, and then the natural logarithm, we obtain the equation from question a).

Question 4.

We consider the optimization problem

$$\min 2x^2 + y^2 + 3z^2 \text{ subject to } \begin{cases} x - y + 2z &= 3 \\ x + y &= 3 \end{cases}$$

(a) The Lagrangian for this problem is given by $\mathcal{L} = 2x^2 + y^2 + 3z^2 - \lambda_1(x - y + 2z) - \lambda_2(x + y)$, and the first order conditions are

$$\begin{aligned} \mathcal{L}'_x &= 4x - \lambda_1 - \lambda_2 = 0 \\ \mathcal{L}'_y &= 2y + \lambda_1 - \lambda_2 = 0 \\ \mathcal{L}'_z &= 6z - 2\lambda_1 = 0 \end{aligned}$$

We solve the first order conditions for x, y, z and get

$$x = \frac{\lambda_1 + \lambda_2}{4}, \quad y = \frac{\lambda_2 - \lambda_1}{2}, \quad z = \frac{\lambda_1}{3}$$

When we substitute these expressions into the two constraints $x - y + 2z = 3$ and $x + y = 3$, we get the equations

$$17\lambda_1 - 3\lambda_2 = 36, \quad -\lambda_1 + 3\lambda_2 = 12$$

Adding the two equations, we get $16\lambda_1 = 48$, or $\lambda_1 = 3$, and the last equation gives $\lambda_2 = 5$. When we substitute this into the expressions for x, y, z we get $(x, y, z) = (2, 1, 1)$. This means that $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ **is the unique point that satisfies the first order conditions and the constraints**. Alternatively, one may observe that the first order conditions and the constraints form a 5×5 linear system. If we substitute $(x, y, z) = (2, 1, 1)$ in this system, we find that $\lambda_1 = 3$ and $\lambda_2 = 5$; hence $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is one solution of the system. To show that this is the only solution, we may check that the determinant of the coefficient matrix is non-zero. We first use some elementary row operations that preserve the determinant:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & 0 & 0 & 17/12 & -1/4 \\ 0 & 0 & 0 & -1/4 & 3/4 \end{vmatrix}$$

Then we see that the determinant is given by $4 \cdot 2 \cdot 6 \cdot (17/4 \cdot 3/4 - 1/4 \cdot 1/4) = 48 \neq 0$.

(b) The bordered Hessian at $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is the matrix

$$B = \begin{pmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Since there are $n = 3$ variables and $m = 2$ constratints, we have to compute the $n - m = 1$ last principal minors; that is, just the determinant $D_5 = |B|$. We first use an elementary row operation to simplify the computation, then develop the determinant along the last column:

$$|B| = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 2 & 0 & -3 & 3 & 0 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix}$$

Then we develop the last determinant along the first row, and get

$$|B| = 2 \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 4 & 0 \\ -1 & 1 & 0 & 2 \\ 2 & 0 & -3 & 3 \end{vmatrix} = 2 \left(\begin{vmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 4 \\ -1 & 1 & 0 \\ 2 & 0 & -3 \end{vmatrix} \right) = 2(10 + 14) = 48$$

Since $|B| = 48 > 0$ has the same sign as $(-1)^m = (-1)^2 = 1$, we conclude that **the point** $(x, y, z) = (2, 1, 1)$ **is a local minimum for** $2x^2 + y^2 + 3z^2$ (among the admissible points). The local minimum value is $f(2, 1, 1) = 8 + 1 + 3 = \mathbf{12}$.

(c) We fix $\lambda_1 = 3$ and $\lambda_2 = 5$, and consider the Lagrangian

$$\mathcal{L}(x, y, z) = 2x^2 + y^2 + 3z^2 - 3(x - y + 2z) - 5(x + y)$$

This function is clearly convex, since the Hessian matrix

$$\mathcal{L}'' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

is positive definite (with eigenvalues $4, 2, 6$). Therefore, the point $(x, y, z) = (2, 1, 1)$ **solves the minimum problem**. The Kuhn-Tucker problem can be reformulated in standard form as

$$\max -(2x^2 + y^2 + 3z^2) \text{ subject to } \begin{cases} -(x - y + 2z) & \leq -3 \\ -(x + y) & \leq -3 \end{cases}$$

Therefore, we see that the Lagrangian of the Kuhn-Tucker problem is

$$-(2x^2 + y^2 + 3z^2) + \lambda_1(x - y + 2z) + \lambda_2(x + y) = -\mathcal{L}$$

and the first order conditions of the Kuhn-Tucker problem are the same as in the original problem. Hence $(x, y, z; \lambda_1, \lambda_2) = (2, 1, 1; 3, 5)$ is still a solution of the first order conditions and the constraints, and $\lambda_1, \lambda_2 \geq 0$ also solves the complementary slackness conditions. When we fix $\lambda_1 = 3$ and $\lambda_2 = 5$, $-\mathcal{L}$ is concave since $\mathcal{L}$ is convex, and this means that $(x, y, z) = (2, 1, 1)$ **also solves the Kuhn-Tucker problem**.