Key Problems

Problem 1.
Find the general solution of the second order differential equations:

a) $y'' + 6y' - 16y = 16t - 22$
b) $y'' + 6y' + 9y = 4e^{-t}$
c) $y'' - 3y' + 2y = 3e^{2t}$
d) $y'' - y = t^2$

Problem 2.
Solve the first order differential equations using the i) superposition principle, and ii) integrating factors.

a) $y' = y - t$
b) $y' + y = e^{-2t}$
c) $y' + 2y = 4t^2$

Problem 3.
Write the systems of differential equations in matrix form and solve them:

a) $\begin{array}{rcl} y_1' &=& 2y_1 + 5y_2 \\ y_2' &=& 5y_1 + 2y_2 \end{array}$
b) $\begin{array}{rcl} y_1' &=& y_2 \\ y_2' &=& 4y_1 + 3y_2 \end{array}$
c) $\begin{array}{rcl} y_1' &=& y_1 + 4y_2 + 3 \\ y_2' &=& y_1 - 2y_2 - 3 \end{array}$

Problem 4.
Solve the systems of differential equations:

a) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
b) $\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$

Problem 5.
Rewrite the differential equation $y''' + 4y'' + y' - 6y = 0$ as a system of first order linear differential equations, and use this to solve the third order differential equation.

Problem 6.
Find the equilibrium states and determine their stability:

a) $y'' + 7y' + 10y = 5$
b) $y'' + y' - 20y = 1$
c) $y''' + 4y'' + y' - 6y = 12$
d) $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$
e) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}$

Exercise Problems

Textbook problems: [E] 7.26 - 7.34, 9.1 - 9.7

Answers to Key Problems

Problem 1.

a) $y = C_1\, e^{-8t} + C_2\, e^{2t} + 1 - t$

b) $y = C_1\, e^{-3t} + C_2 t e^{-3t} + e^{-t}$

c) $y = C_1\, e^{2t} + C_2 e^{t} + 3te^{2t}$

d) $y = C_1\, e^{t} + C_2\, e^{-t} - t^2 - 2$

Problem 2.

a) $y = C\, e^{t} + t + 1$

b) $y = C\, e^{-t} - e^{-2t}$

c) $y = C\, e^{-2t} + 2t^2 - 2t + 1$

Problem 3.

a) $\mathbf{y}' = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$

b) $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = C_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

c) $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \quad \mathbf{y} = C_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 4.

a) $\mathbf{y} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-4t} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-6t}$

b) $\mathbf{y} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}$

Problem 5.

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3t}$$

Problem 6.

a) $\mathbf{y}_e = 1/2$ is globally asymptotically stable

b) $\mathbf{y}_e = -1/20$ is unstable

c) $\mathbf{y}_e = -2$ is unstable

d) $\mathbf{y}_e = (1, -1)$ is unstable

e) $\mathbf{y}_e = (0,0,0)$ is globally asymptotically stable