	Plan	Textbook
1	Linear difference equations	[E] 8.1 - 8.2, 8.4
2	Systems of linear first order difference equations	[E] 9.4
3	Equilibrium states and stability	[E] 8.3, 9.5
	Textbook problems	[E] 8.1 - 8.9, 9.8

Plenary Session 4: Mon 10/11 at 16-19 (Lecture 10-12)

Lecture 13: Going thr. $\boxed{\text{Final } 11/2024}$

Review:

L10: First order — separable / linear / exact

Eq. states / stability

L11: Second order linear
Systems

Eq. states / stability:

a) Second order: $y'' + ay' + by = c$

Eq. states: $y' = 0, y'' = 0$ $\quad by = c$

$y_e = c/b \quad (b \neq 0)$

Stability: $r_1, r_2 < 0$ stable

otherwise unstable

Typical ex: $y'' + 5y' + 4y = 12$

$y_e = \underline{3}$

$r^2 + 5r + 4 = 0$

$(r+1)(r+4) = 0$

$r = -1, r = -4$

$y_e = 3$ is stable (and gl. asymp. stable)

$$y = \underline{C_1 e^{-t} + C_2 e^{-4t} + 3}$$

b) Systems: $\mathbf{y}' = A \cdot \mathbf{y} + \mathbf{b}$

Eq. states: $A\mathbf{y} + \mathbf{b} = \mathbf{0}$

$A \cdot \mathbf{y} = -\mathbf{b}$

Stability:

All eigenvalues of A negative

$\Rightarrow \mathbf{y}_e$ stable (and gl. as. stable)

Otherwise: unstable

$$\mathbf{y} = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + C_n \mathbf{v}_n e^{\lambda_n t} + \mathbf{y}_e$$

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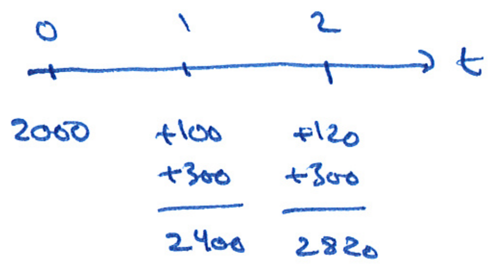
Ex: $y_{t+1} = 1.05 y_t + 300$, $y_0 = 2000$

Find: $y_0, y_1, y_2, \dots$ (discrete time)

$y_1 = 1.05 \cdot 2000 + 300 = 2400$

$y_2 = 1.05 \cdot 2400 + 300 = 2820$

$\vdots$

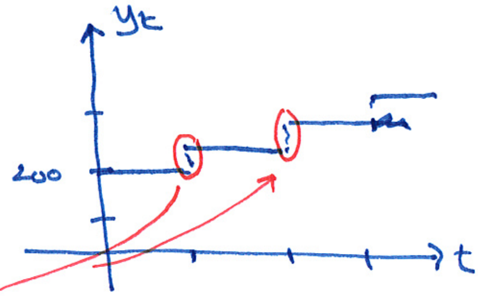


$y_{50} = ?$ Closed formula for y_t

Interpretation: $\Delta y_t = y_{t+1} - y_t$

$y_{t+1} = 1.05 y_t + 300$

$y_{t+1} - y_t = \underline{0.05 y_t + 300}$



Solution: $y_{t+1} - 1.05 y_t = 300$, $y_0 = 2000$

$y_t = y_t^h + y_t^p = \underline{C \cdot 1.05^t - 6000}$

Compare: $y' + ay = b(t)$

y_t^h: $y_{t+1} - 1.05 y_t = 0$ $\boxed{\begin{array}{l} y_{t+1} \to r \\ y_t \to 1 \end{array}}$

Char. eqn: $r - 1.05 = 0$

$\underline{r = 1.05}$ $y_t^h = C \cdot r^t = \underline{C \cdot 1.05^t}$

$y_h = C \cdot e^{rt}$

y_t^p: $y_{t+1} - 1.05 y_t = 300$

$A - 1.05 A = 300$ $\left\{ \begin{array}{l} y_t = A \\ y_{t+1} = A \end{array} \right.$

$-0.05 A = 300$

$A = \dfrac{300}{-0.05} = -6000$ $y_t^p = -6000$

$y_0 = 2000$: $C \cdot 1.05^0 - 6000 = 2000$

$C - 6000 = 2000$

$C = 8000$

$\to y_t = \underline{\underline{8000 \cdot 1.05^t - 6000}}$

$y_{50} = 8000 \cdot 1.05^{50} - 6000$

$\simeq \underline{85.739}$

① Linear difference equations

$$\boxed{\begin{array}{ll} \text{First order:} & y_{t+1} + a y_t = b_t \\ \text{Second order:} & y_{t+2} + a y_{t+1} + b y_t = c_t \end{array}}$$

a const, b_t: expr in t

a, b const, c_t: expr in t

Superposition: $y_t = y_t^h + y_t^p$ where y_t^h: general solution of the homogeneous eqn.

To find y_t^h, we use the char eqn. y_t^p: particular solution of the original eqn.

$y_{t+1} + a y_t = 0$

$r + a = 0 \quad r = -a \quad y_t^h = c \cdot (-a)^t$

$y_{t+2} \rightsquigarrow r^2$

$y_{t+1} \rightsquigarrow r$

$y_t \longrightarrow 1$

$y_{t+2} + a y_{t+1} + b y_t = 0$

$r^2 + ar + b = 0$

$r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

$r_1 \neq r_2$: $\quad y_t^h = \underline{C_1 \cdot r_1^t + C_2 \cdot r_2^t}$

$r_1 = r_2$: $r = -a/2$ $\quad y_t^h = \underline{C_1 \cdot r^t + C_2 \cdot t r^t}$

$= (C_1 + C_2 t) \cdot r^t$

no real roots: $\quad y_t^h = \ldots$

Difference and second order diff:

$\Delta y_t = \underline{y_{t+1} - y_t}$

$\Delta y_{t+1} - \Delta y_t = y_{t+2} - y_{t+1} - (y_{t+1} - y_t)$

$= \underline{y_{t+2} - 2y_{t+1} + y_t}$

To find y_t^p:

Guess y_t based on the right-hand side, with parameters. Verify in the difference eqn.

If necessary, multiply your guess with t.

Ex: $y_{t+1} - y_t = \underline{1}$ $\quad y_t = y_t^h + y_t^p = \underline{\underline{C + t}}$

y_t^h: $r - 1 = 0 \quad r = 1 \quad y_t^h = C \cdot 1^t = \underline{C}$

y_t^p: $\left.\begin{array}{l} y_t = A \\ y_{t+1} = A \end{array}\right\}$ $A - A = 1$

$0 = 1$ $\underline{\text{impossible}}$

$\left.\begin{array}{l} y_t = \underline{At} \\ y_{t+1} = A(t+1) = At + A \end{array}\right\}$ $(\cancel{At} + A) - (\cancel{At}) = 1$

$A = 1$

$y_t^p = \underline{t}$

Ex: $y_{t+2} - 3y_{t+1} + 2y_t = \underline{t}$ $\quad y_t = y_t^h + y_t^p = \underline{\underline{C_1 + C_2 \cdot 2^t - \tfrac{1}{2}t^2 - \tfrac{1}{2}t}}$

$\underline{y_t^h}$: $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0 \quad r_1 = 1, r_2 = 2 \quad y_t^h = C_1 \cdot 1^t + C_2 \cdot 2^t$

$= \underline{C_1 + C_2 \cdot 2^t}$

$\underline{y_t^p}$: $y_t = \underline{At + B}$

$y_{t+1} = A(t+1) + B$

$= \underline{At + A + B}$

$y_{t+2} = A(t+2) + B$

$= \underline{At + 2A + B}$

$(\underline{At} + \underline{2A+B}) - 3(\underline{At} + \underline{A+B}) + 2(\underline{At+B}) = t$

$(\underline{At - 3At + 2At}) + (-A) = t$

$-A = t$ impossible

$y_t = \underline{At^2 + Bt}$

$y_{t+1} = A(t+1)^2 + B(t+1)$

$= \underline{At^2 + 2At + Bt + A + B}$

$y_{t+2} = A(t+2)^2 + B(t+2)$

$= \underline{At^2 + 4At + Bt + 4A + 2B}$

$(\cancel{At^2} + \underline{4At} + \cancel{Bt} + 4A + 2B)$

$-3(\cancel{At^2} + \underline{2At} + \cancel{Bt} + A + B)$

$+2(\cancel{At^2} + \cancel{Bt}) = t$

$(-2A)t + (\underline{4A + 2B - 3A - 3B}) = t$

$-2A = 1 \qquad A - B = 0$

$\underline{A = -1/2} \qquad B = A = \underline{-1/2}$

$y_t^p = -\tfrac{1}{2}t^2 - \tfrac{1}{2}t$

2) Systems of linear difference equations

$\underline{y}_{t+1} = A \cdot \underline{y}_t + \underline{b}$

Ex: $\begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \end{pmatrix} = \begin{pmatrix} 0.76 & 0.12 \\ 0.24 & 0.88 \end{pmatrix} \cdot \begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix}$

$\underline{v}_{t+1} = A \cdot \underline{v}_t$

Solution:

$\underline{y}_t = \underline{y}_t^h + \underline{y}_t^p$

i) If A is diagonalizable, then

$$\underline{y}_t^h = c_1 \cdot \underline{v}_1 \cdot \lambda_1^t + c_2 \cdot \underline{v}_2 \cdot \lambda_2^t + \dots + c_n \cdot \underline{v}_n \lambda_n^t$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A, $\underline{v}_1, \dots, \underline{v}_n$ the corresponding eigenvectors.

ii) $\underline{y}_t^p = \underline{y}_e$: Eq. state = constant solution.

$\underline{y}_{t+1} = \underline{y}_t$: $\underline{y}_t = A\underline{y}_t + \underline{b}$

$-\underline{b} = A\underline{y}_t - \underline{y}_t$

$A\underline{y}_t - I\underline{y}_t = -\underline{b}$

$(A - I)\underline{y}_t = -\underline{b}$

$\underline{y}_e$ is soln of

$$\boxed{(A - I)\underline{y}_e = -\underline{b}}$$

Ex: $\underline{y}_{t+1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \underline{y}_t + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda_1 = 1, \lambda_2 = 3$

E_1: $\left(\begin{array}{cc} 1 & -1 \\ \hline -1 & 1 \end{array}\right)$ $x - y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_3: $\left(\begin{array}{cc} -1 & -1 \\ \hline -1 & -1 \end{array}\right)$ $-x - y = 0$, y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{y}_t^h = c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 1^t + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t$

$= c_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t$ //

$\underline{y}_t^p = \underline{y}_e$: $\underline{y}_e = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \underline{y}_e + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$-\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \underline{y}_e - I \underline{y}_e = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \underline{y}_e$

$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ -1 & 1 & +3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 0 & 0 \end{array}\right)$ $x - y = -3$, y free

$y = 0$: $\underline{y}_e = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \Rightarrow \underline{y}_t^p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

Solution: $\underline{\underline{\underline{y}_t = C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot 3^t + \begin{pmatrix} -3 \\ 0 \end{pmatrix}}}$

$y_{1,t} = C_1 - C_2 \cdot 3^t - 3$

$y_{2,t} = C_1 + C_2 \cdot 3^t$

③ Eq. states and stability

Eq. states = constant solutions!

Ex: $y_{t+1} = 1.05 y_t + 300$

Eq. state: $y_{t+1} = y_t = y_e$

$y_e = 1.05 y_e + 300$

$y_e - 1.05 y_e = 300$

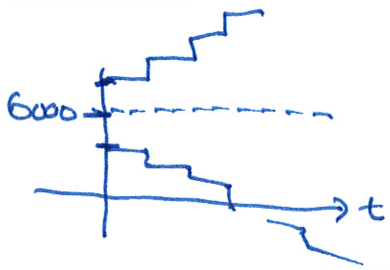
$-0.05 y_e = 300$

$y_e = \frac{300}{-0.05} = \underline{-6000}$

Eq. state: $y_e = -6000$

General solution: $y_t = C \cdot 1.05^t - 6000$

$y_0 = C - 6000$



$y_e = -6000$ is unstable

since $y_t \to \pm\infty$ if $y_0 \neq y_e$

y_e stable if $-1 < r < 1$ | $-1 < r_1, r_2 < 1$ | $-1 < \lambda_1, \lambda_2, \ldots < 1$
unstable otherwise

first order linear | second order linear | systems