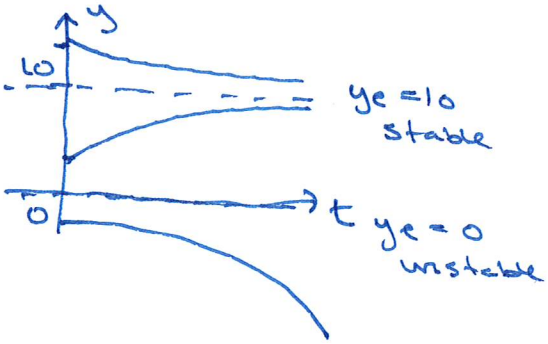
	Plan	Textbook
1	Linear second order differential equations	[E] 7.9 - 7.10
2	Systems of differential equations	[E] 9.1 - 9.3
	Textbook problems	[E] 7.26 - 7.34, 9.1 - 9.7

Review: First order differential eqn's $y' = F(t,y)$ - seperable - linear - exact

Eq. states and Stability: $y' = F(y)$

Ex: $y' = 5y(1 - y/10)$

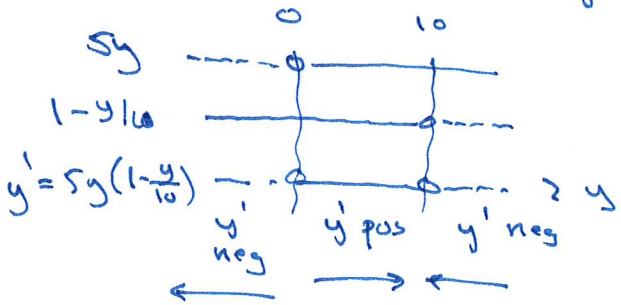


Eq. states: $F(y) = 0 \Leftrightarrow y' = 0$

$5y(1 - y/10) = 0$

$y_e = 0$ or $y_e = 10$

Stability: Sign diagram for y'

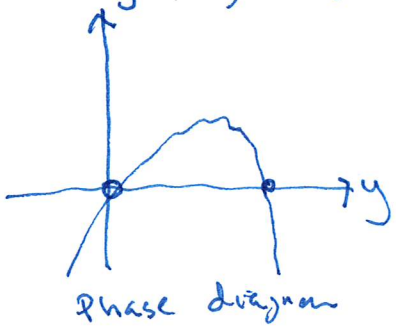


stable eq. state gl. asympt. stable

unstable eq. state

stable, not gl. asymptotically stable

$y' = 5y - \frac{1}{2}y^2$



Phase diagram

Stability Thm:

$F'(y_e) < 0 \Rightarrow y_e$ stable

$F'(y_e) > 0 \Rightarrow y_e$ unstable

(2) Linear second order diff. eqn's

Ex: $y'' = 6t$

$y' = \int 6t\,dt = 3t^2 + C$

$y = \int 3t^2 + C\,dt = t^3 + Ct + D$

Second order diff. eqn: general solution depends on two undetermined coeff.

Defn A second order diff. eqn. is called linear with constant coeff. if it can be written

$$y'' + ay' + by = h(t)$$

a, b: constants; $h(t)$: expr. in t

It is called homogeneous if $h(t) = 0$, inhomogeneous otherwise.

i) Homogeneous case: $y'' + ay' + by = 0$

"Magic" trick: Char. eqn $\boxed{r^2 + ar + b = 0}$

Char roots: $r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

$y'' \to r^2$, $y' \to r$, $y \to 1$

Cases: i) Two distinct roots $r_1 \neq r_2$: $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

ii) One root $r = -a/2$ of mult 2: $y = C_1 e^{rt} + C_2 t e^{rt}$

iii) No (real) roots

$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \Rightarrow \alpha = \frac{-a}{2}$, $\beta = \frac{\sqrt{4b - a^2}}{2}$

$y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

Ex: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$ $r = 1$, $r = 2$ $\Rightarrow$ $y = C_1 e^{1t} + C_2 e^{2t}$

Ex: $y' - 5y = 0$

$r - 5 = 0$ $r = 5$ $\Rightarrow$ $y = C \cdot e^{5t}$

Ex: $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0 \quad (r-2)^2 = 0$

$r_1 = r_2 = 2$

$y = C_1 \cdot e^{2t} + C_2 \cdot te^{2t}$

$= (C_1 + C_2 t)e^{2t}$

Explanation: $y'' + ay' + by = 0$

$r^2 e^{rt} + a(re^{rt}) + be^{rt} = 0$

$e^{rt}(r^2 + ar + b) = 0$

Try: $e^{rt} = y$

$y' = e^{rt} \cdot r$

$y'' = e^{rt} \cdot r^2$

Char. eqn.: $r^2 + ar + b = 0$

Concl.: $y = e^{rt}$ is a solution of the diff. eqn. $\iff$ r is a root in $r^2 + ar + b = 0$

ii) **Inhomogeneous case:** $y'' + ay' + by = h(t)$

Superposition principle: The general solution is $y = y_h + y_p$,

where

- y_h: general solution of the homogeneous equation $\quad y'' + ay' + by = 0$
- y_p: particular solution of the inhomogeneous eqn. $\quad y'' + ay' + by = h(t)$

Ex: $y'' - 4y' + 3y = 3e^{2t}$

$y = y_h + y_p = C_1 e^t + C_2 e^{3t} + 3e^{2t}$

y_h: $y'' - 4y' + 3y = 0$

$r^2 - 4r + 3 = 0 \quad r_1 = 1, r_2 = 3$ $\Big\}$ $y_h = C_1 e^t + C_2 e^{3t}$

y_p: $y'' - 4y' + 3y = 3e^{2t}$

$4Ae^{2t} - 4(2Ae^{2t}) + 3(Ae^{2t}) = 3e^{2t}$

$(4A - 8A + 3A)e^{2t} = 3e^{2t}$

$-A = 3 \quad A = -3$

$h(t) = 3e^{2t}$ $\rightsquigarrow$ $y = A \cdot e^{2t}$ (A const)

$h'(t) = 3 \cdot e^{2t} \cdot 2$ $\quad y' = A e^{2t} \cdot 2 = 2A \cdot e^{2t}$

$h''(t) = 3 \cdot e^{2t} \cdot 2 \cdot 2$ $\quad y'' = 2A \cdot e^{2t} \cdot 2 = 4A \cdot e^{2t}$

$y_p = -3e^{2t}$

Method for finding y_p: Variation of parameters

$\boxed{y'' + ay' + by = h(t)}$

Look at $h(t)$, choose a guess of the same form, but with parameters that can be varied.

Ex: $y'' + 4y' + 3y = 3e^{2t} \rightsquigarrow y = C_1 e^{t} + C_2 e^{3t} + 3e^{2t}$

$y = Ae^{2t}$

Hints: i) It can be useful to look at h but also h', h''.
ii) If your initial guess does not work, try to multiply it with t.

Ex: $y'' - 4y' + 3y = 2e^{3t} \rightsquigarrow y = C_1 e^{t} + C_2 e^{3t} + te^{3t}$

Guess: $h = 2e^{3t} \rightsquigarrow y = Ae^{3t}$

$h' = 2e^{3t} \cdot 3 \quad y' = 3Ae^{3t}$

$h'' = 2e^{3t} \cdot 3 \cdot 3 \quad y'' = 9Ae^{3t}$

$(9Ae^{3t}) - 4(3Ae^{3t}) + 3(Ae^{3t}) = 2e^{3t}$

$(9A - 12A + 3A)e^{3t} = 2e^{3t}$

$0A \cdot e^{3t} = 2e^{3t}$

it does not work!

Try: $y = Ae^{3t} \cdot t = Ate^{3t}$

$y' = A \cdot e^{3t} + At \cdot e^{3t} \cdot 3$

$= Ae^{3t} + 3Ate^{3t}$

$= (3At + A)e^{3t}$

$y'' = 3A \cdot e^{3t} + (3At + A) \cdot e^{3t} \cdot 3$

$= (3A + 9At + 3A)e^{3t}$

$= (9At + 6A)e^{3t}$

$(9At + 6A)e^{3t} - 4(3At + A)e^{3t} + 3(Ate^{3t}) = 2e^{3t}$

$(\cancel{9At} - \cancel{12At} + \cancel{3At})e^{3t} + (6A - 4A)e^{3t} = 2e^{3t}$

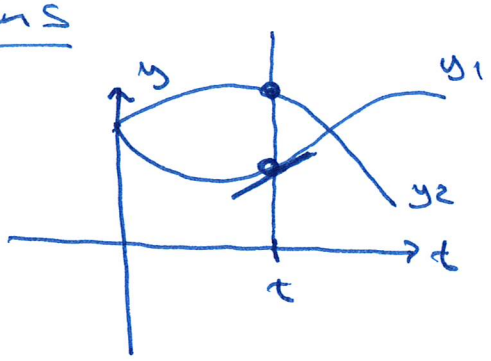
$2A \cdot e^{3t} = 2 \cdot e^{3t}$

$2A = 2$

$A = 1$

$y_p = te^{3t}$

(2) Systems of differential equations



Ex:
$$y_1' = y_1 + 3y_2 - 7$$
$$y_2' = 2y_1 - y_2 + 3$$
coupled

Ex:
$$y_1' = 3y_1 \qquad y_1' - 3y_1 = 0 \qquad y_1 = C_1 e^{3t}$$
$$y_2' = 5y_2 \qquad y_2' - 5y_2 = 0 \qquad y_2 = C_2 e^{5t}$$
decoupled

Ex:
$$\left.\begin{aligned} y_1' &= y_1 + 2y_2 - 4 \\ y_2' &= 2y_1 + y_2 - 5 \end{aligned}\right\} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \underline{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$
$$\boxed{\underline{y}' = A\underline{y} + \underline{b}}$$
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -4 \\ -5 \end{pmatrix}$$

Def A system of linear first order differential equations that are autonomous can be written:

$$\boxed{\underline{y}' = A\underline{y} + \underline{b}} \quad \text{with} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$
A: $n \times n$-matrix
$\underline{b}$: n-vector

It is called homogeneous if $\underline{b} = \underline{0}$, and inhomogeneous otherwise

$$\underline{y}' = A\underline{y} \Leftrightarrow \underline{y}' - A\underline{y} = \underline{0}$$

i) Homogeneous case: $\underline{y}' = A\underline{y}$ A n×n-matrix

Result: If A is diagonalizable, with eigenvalues/eigenvectors $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{pmatrix} \qquad P = \left| \underline{v}_1 \right| \underline{v}_2 \left| \ldots \right| \underline{v}_n \right)$$

then

$$\underline{y} = c_1 \cdot \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + \ldots + c_n \underline{v}_n e^{\lambda_n t}$$

is the general solution of the system of diff.-eqns.

Ex: $\left.\begin{aligned} y_1' &= y_1 + 2y_2 \\ y_2' &= 2y_1 + y_2 \end{aligned}\right\} \rightarrow \underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \underline{y} \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Eigenvalues: $\lambda^2 - 2\lambda + (-3) = 0$

$\lambda_1 = -1, \ \lambda_2 = 3$

Eigenvectors: $\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} \textcircled{2} & 2 \\ 0 & 0 \end{pmatrix}$ $2x + 2y = 0$, y free

$x = -y \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Base: $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} \textcircled{-2} & 2 \\ 0 & 0 \end{pmatrix}$ $-2x + 2y = 0$, y free

$x = y \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Base: $\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \underline{y} = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$ is the general solution

$y_1 = -c_1 e^{-t} + c_2 e^{3t}$

$y_2 = c_1 e^{-t} + c_2 e^{3t}$

ii) Inhomogeneous case: $\underline{y}' = A\cdot\underline{y} + \underline{b}$

Superposition principle $\underline{y} = \underline{y}_h + \underline{y}_p$

$\underline{y}_h$: general solution of the homogeneous eqn $\underline{y}' = A\underline{y}$

$\underline{y}_p$: particular solution of $\underline{y}' = A\underline{y} + \underline{b}$ ← Look for constant solutions, i.e. $\underline{y}' = \underline{0}$

Ex: $y_1' = y_1 + 2y_2 - 4$
$y_2' = 2y_1 + y_2 - 5$

$\underline{y} = \underline{y}_h + \underline{y}_p$
$= c_1\cdot\begin{pmatrix}-1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + \begin{pmatrix}2\\1\end{pmatrix}$

$\underline{y}_p$: $y_1 + 2y_2 - 4 = 0$
$2y_1 + y_2 - 5 = 0$

$A\cdot\underline{y} + \underline{b} = \underline{0} \Rightarrow \boxed{A\cdot\underline{y} = -\underline{b}}$

$\underline{y}_p = \begin{pmatrix}2\\1\end{pmatrix}$

$\left(\begin{array}{cc|c}1 & 2 & 4\\ 2 & 1 & 5\end{array}\right) \xrightarrow{-2} \left(\begin{array}{cc|c}1 & 2 & 4\\ 0 & -3 & -3\end{array}\right)$

$y_1 + 2y_2 = 4$
$-3y_2 = -3$
$y_1 + 2 = 4$
$y_2 = 1$
$y_1 = 2$

Ex: $y'' - 3y' + 2y = 0 \rightarrow \boxed{\begin{pmatrix}y_1'\\y_2'\end{pmatrix} = \begin{pmatrix}0 & 1\\-2 & 3\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}}$

$\boxed{y_1 = y,\ y_2 = y'}$

$y_1' = y_2$
$y_2' = y'' = 3y' - 2y = 3y_2 - 2y_1$

$r^2 - 3r + 2 = 0$

$\begin{vmatrix}-\lambda & 1\\ -2 & 3-\lambda\end{vmatrix} = 0$

$-\lambda(3-\lambda) + 2 = 0$
$\lambda^2 - 3\lambda + 2 = 0$

Explanation: $\underline{y}' = A\underline{y}$

Change of variables: $\underline{y} = P \cdot \underline{u}$, $\underline{y}' = P \cdot \underline{u}'$

$P\underline{u}' = A \cdot (P\underline{u})$

$\underline{u}' = P^{-1}AP \cdot \underline{u}$

$\underline{u}' = D \cdot \underline{u}$

Assume A diag: $P^{-1}AP = D$, $AP = PD$

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

decoupled

$u_1' = \lambda_1 u_1$
$u_2' = \lambda_2 u_2$
$\vdots$
$u_n' = \lambda_n u_n$

$u_1 = c_1 e^{\lambda_1 t}$
$u_2 = c_2 e^{\lambda_2 t}$
$\vdots$
$u_n = c_n e^{\lambda_n t}$

$$\underline{y} = P \cdot \underline{u} = P \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= P \cdot c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_1 t} + P c_2 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} e^{\lambda_2 t} + \cdots$$

$$= c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + \cdots$$

$P \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underline{v}_1$, $P \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \underline{v}_2$, ...