Key Problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ:

a) $A = \begin{pmatrix} 5 & 9 \\ 9 & 5 \end{pmatrix}$ b) $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & -4 \\ 3 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ e) $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ f) $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Problem 2.
For the matrix A in Problem 1 a) - f), determine whether A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible. Hint: It is not necessary to compute P^{-1} to answer this question.

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.
Use eigenvalues and eigenvectors of A to determine the limit of A^m when $m \to \infty$.

a) $A = \begin{pmatrix} 0.40 & 0.15 \\ 0.60 & 0.85 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$

Problem 5.
When A is a 3×3 matrix, the characteristic equation of A can be written as $-\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0$, where $c_1 = \mathrm{tr}(A)$, $c_2 = M_{12,12} + M_{23,23} + M_{13,13}$ and $c_3 = \det(A)$. Use this formula to find the characteristic equation and the eigenvalues of the following matrices:

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 0 & 4 & 7 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

Optional task: Prove the formula by writing down the characteristic equation of a 3×3 matrix $A = (a_{ij})$ with general coefficients.

Exercise Problems

Problems from the textbook:	[E] 4.1 - 4.7
Exam problems:	[Midterm 10/2018] Question 1-6
	[Midterm 10/2022] Question 3,6,8

Answers to Key Problems

Problem 1.

a. Eigenvalues $\lambda_1 = -4$, $\lambda_2 = 14$ and eigenvectors $E_{-4} = \text{span}(\mathbf{v}_1)$ and $E_{14} = \text{span}(\mathbf{v}_2)$, where $\mathbf{v}_1 = (-1{,}1)$ and $\mathbf{v}_2 = (1{,}1)$.

b. Eigenvalues $\lambda_1 = \lambda_2 = 3$ and eigenvectors $E_3 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = (1{,}1)$.

c. No eigenvalues or eigenvectors.

d. Eigenvalues $\lambda_1 = \lambda_2 = 4$, $\lambda_3 = 2$ and eigenvectors $E_4 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_2 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = (0{,}1{,}0)$, $\mathbf{v}_2 = (1{,}0{,}1)$, and $\mathbf{v}_3 = (-1{,}0{,}1)$.

e. Eigenvalues $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 2$ and eigenvectors $E_{-1} = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_2 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = (-1{,}1{,}0)$, $\mathbf{v}_2 = (-1{,}0{,}1)$, and $\mathbf{v}_3 = (1{,}1{,}1)$.

f. Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and eigenvectors $E_0 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = (1{,}0{,}0)$.

Problem 2.

a) Yes, with $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 0 \\ 0 & 14 \end{pmatrix}$

b) No

c) No

d) Yes, with $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

e) Yes, with $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

f) No

Problem 3.
The eigenvalues of A are $\lambda_1 = \lambda_2 = 5$, $\lambda_3 = -1$ and $\lambda_4 = -3$.

Problem 4.

a) $A^m \to \begin{pmatrix} 1/5 & 1/5 \\ 4/5 & 4/5 \end{pmatrix}$ as $m \to \infty$

b) $A^m \to \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$ as $m \to \infty$

Problem 5.

a) $-\lambda^3 + 6\lambda^2 - 4\lambda = 0$, $\lambda = 0$ or $\lambda = 3 \pm \sqrt{5}$

b) $-\lambda^3 + 9\lambda^2 - 18\lambda + 8 = 0$, $\lambda = 2$ or $\lambda = (7 \pm \sqrt{33})/2$

c) $-\lambda^3 = 0$, $\lambda = 0$ (multiplicity 3)