	Plan	
1	Key Problems:	11.3c, 12.2ab, 12.3, 12.4bc, 12.5be
2	Final exams:	01/2024 1d, 3d 11/2019 4bc 11/2018 2c,5

① Key problems

11.3 c) $y' + 2y = 4t^2$

Superposition: $y = y_h + y_p = \underline{Ce^{-2t} + 2t^2 - 2t + 1}$

y_h: $y' + 2y = 0$

$r + 2 = 0 \quad r = -2 \quad y_h = \underline{Ce^{-2t}}$

y_p: $y' + 2y = 4t^2$

$$\left.\begin{aligned} y &= At^2 + Bt + C \\ y' &= 2At + B \end{aligned}\right\} \quad (2At + B) + 2(At^2 + Bt + C) = 4t^2$$

$$\underbrace{(2A)}_{=4}t^2 + \underbrace{(2A + 2B)}_{=0}t + \underbrace{(B + 2C)}_{=0} = 4t^2 + 0t + 0$$

$y_p = \underline{2t^2 - 2t + 1}$ $\quad \underline{A = 2} \quad \underline{B = -2} \quad \underline{C = 1}$

Int. factor: $y' + 2y = 4t^2 \,|\cdot e^{2t} \quad a(t) = 2 \quad \int a(t)\,dt = \int 2\,dt = 2t + C$

$(y \cdot e^{2t})' = 4t^2 e^{2t} \quad \Rightarrow u = \underline{e^{2t}}$

$$y \cdot e^{2t} = \int \underbrace{4t^2}_{v} \underbrace{e^{2t}}_{u'}\,dt = \frac{1}{2}e^{2t} \cdot 4t^2 - \int \underbrace{4t}_{v} \underbrace{e^{2t}}_{u'}\,dt$$

$v' = 8t \quad u = \frac{1}{2}e^{2t} \qquad v' = 4 \quad u = \frac{1}{2}e^{2t}$

$$= 2t^2 e^{2t} - \left(4t \cdot \tfrac{1}{2}e^{2t} - \int 2e^{2t}\,dt\right.$$

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$$\frac{y\cdot e^{2t}}{e^{2t}} = \frac{2t^2e^{2t} - 2te^{2t} + e^{2t} + C}{e^{2t}}$$

$$\underline{\underline{y = 2t^2 - 2t + 1 + Ce^{-2t}}}$$

12.2b) $y' = A\mathbf{y}$ $\quad A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix}$, $\quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$

$$\mathbf{y} = C_1 \cdot \mathbf{v_1} e^{\lambda_1 t} + C_2 \mathbf{v_2} e^{\lambda_2 t} + C_3 \mathbf{v_3} e^{\lambda_3 t}$$

$$= \underline{C_1 \cdot \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + C_3 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}}$$

Eigenvalues:

$-\lambda^3 + 5\lambda^2 - c_2 \cdot \lambda + \det(A) = 0$

$c_2 = 5 + 2 - 1 = 6$

$|A| = 1(2) + 2\cdot(-1) = 0$

$-\lambda^3 + 5\lambda^2 - 6\lambda + 0 = 0$

$-\lambda(\lambda^2 - 5\lambda + 6) = 0$

$-\lambda(\lambda - 2)(\lambda - 3) = 0$

$\underline{\lambda_1 = 0}$, $\underline{\lambda_2 = 2}$, $\underline{\lambda_3 = 3}$

Eigenvectors:

E_0: $\begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ \cancel{0} & \cancel{-10} & \cancel{-2} \end{pmatrix}$

$x = -2z/5$

$x + 3(-z/5) + z = 0$

$5y + z = 0 \quad y = \underline{-z/5}$

z free

$\Rightarrow \frac{z}{5} \cdot \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} \Rightarrow \mathbf{v_1} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix}$

E_2: $\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 3 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ \cancel{0} & \cancel{-1} & \cancel{-1} \end{pmatrix}$

$x = 0$

$y = -z$

z free

$z\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{v_2} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\underline{E_3}$: $\begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ 3 & -1 & -2 \end{pmatrix} \begin{matrix} \\ -1 \\ 3 \end{matrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -1 \\ \cancel{0} & \cancel{2} & \cancel{1} \end{pmatrix}$ $\begin{matrix} -x+y+z=0 & x = z/2 \\ -2y-z=0 & y=-z/2 \\ z \text{ free} & \end{matrix} \Rightarrow \frac{z}{2}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$y(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$: $c_1\begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + c_2 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -2 & 0 & 1 & -1 \\ -1 & -1 & -1 & -3 \\ 5 & 1 & 2 & 8 \end{array}\right) \updownarrow \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ -2 & 0 & 1 & -1 \\ 5 & 1 & 2 & 8 \end{array}\right) \begin{matrix} \\ -2 \\ 5 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ 0 & 2 & 3 & 5 \\ 0 & -4 & -3 & -7 \end{array}\right) \begin{matrix} \\ \\ 2 \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|c} -1 & -1 & -1 & -3 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 3 & 3 \end{array}\right) \qquad \begin{matrix} -c_1 - 1 - 1 = -3 & \underline{c_1 = 1} \\ 2c_2 + 3(1) = 5 & \underline{c_2 = 1} \\ 3c_3 = 3 & \underline{c_3 = 1} \end{matrix}$$

$\underline{\text{Solution:}}$ $$\underline{\underline{y = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}}}$$

$\underline{12.3}$ $y''' + 4y'' + y' - 6y = 0$

$y_1 = y$

$y_2 = y'$

$y_3 = y''$

$$y''' = 6y - y' - 4y'' = 6y_1 - y_2 - 4y_3$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ 6y_1 - y_2 - 4y_3 \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = c_1 \underline{v_1} e^{\lambda_1 t} + c_2 \underline{v_2} e^{\lambda_2 t} + c_3 \underline{v_3} e^{\lambda_3 t}$$

Eigenvalues: $|A-\lambda I|=0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -1 & -4-\lambda \end{vmatrix} = -\lambda(-\lambda(-4-\lambda)+1)-1(-6)=0$$

$$-\lambda(\lambda^2+4\lambda+1)+6=0$$

$$-\lambda^3-4\lambda^2-\lambda+6=0$$

$\lambda=1$ is a solution

$$(\lambda-1)\cdot(-\lambda^2-5\lambda-6)=0$$

$$-(\lambda-1)(\lambda^2+5\lambda+6)=0$$

$$-(\lambda-1)(\lambda+2)(\lambda+3)=0$$

$$\lambda_1=1,\ \lambda_2=-2,\ \lambda_3=-3$$

$$(-\lambda^3-4\lambda^2-\lambda+6):(\lambda-1)=-\lambda^2-5\lambda-6$$
$$\begin{array}{l} -\lambda^3+\lambda^2 \\ \hline -5\lambda^2 \\ -5\lambda^2+5\lambda \\ \hline -6\lambda \\ -6\lambda+6 \\ \hline 0 \end{array}$$

E_1: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ \cancel{6} & \cancel{-1} & \cancel{-5} \end{pmatrix}$ $\quad x=z,\ y=z,\ z$ free $\quad z\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad \underline{v_1=\begin{pmatrix}1\\1\\1\end{pmatrix}}$

E_3: $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ \cancel{6} & \cancel{-1} & \cancel{-1} \end{pmatrix}$ $\quad 3x-z/3=0\ \ x=z/9,\ \ 3y+z=0\ \ y=-z/3,\ \ z$ free $\quad \frac{z}{9}\begin{pmatrix}1\\-3\\9\end{pmatrix}$ $\quad \underline{v_3=\begin{pmatrix}1\\-3\\9\end{pmatrix}}$

E_{-2}: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ \cancel{6} & \cancel{-1} & \cancel{-2} \end{pmatrix}$ $\quad 2x+(-z/2)=0\ \ x=z/4,\ \ 2y+z=0\ \ y=-z/2,\ \ z$ free $\quad \frac{z}{4}\begin{pmatrix}1\\-2\\4\end{pmatrix}$ $\quad \underline{v_2=\begin{pmatrix}1\\-2\\4\end{pmatrix}}$

$$\mathbf{y}=\begin{pmatrix}y\\y'\\y''\end{pmatrix}=\underline{\underline{c_1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}e^{t}+c_2\begin{pmatrix}1\\-2\\4\end{pmatrix}e^{-2t}+c_3\begin{pmatrix}1\\-3\\9\end{pmatrix}e^{-3t}}}$$

$$y=c_1e^{t}+c_2e^{-2t}+c_3e^{-3t}\ //$$

Alt without system:

$$y'''+4y''+y'-6y=0$$
$$r^3+4r^2+r-6=0$$
$$r_1=1,\ r_2=-2,\ r_3=-3$$

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12.4 $y = \underline{3e^{-2t} - 5e^{t}} + \underline{12e^{-3t}}$

a) $y'' + ay' + by = c(t) = 0 \quad y = \underbrace{c_1 e^{r_1 t} + c_2 e^{r_2 t}}_{y_h} + y_p$

$(r-(-2))(r+1) = 0$

$(r+2)(r-1) = 0$

$r^2 + r - 2 = 0 \implies y'' + y' - 2y = c(t)$

$(108 + (-36) - 2 \cdot 12)\, e^{-3t} = c(t)$

$y_p = 12e^{-3t}$

$y' = -36e^{-3t}$ $\qquad c(t) = \underline{48e^{-3t}}$

$y'' = +108e^{-3t} \implies \underline{y'' + y' - 2y = 48e^{-3t}}$

b) $y_h = y = 3e^{-2t} - 5e^{t} + 12e^{-3t} \qquad r_1 = -2,\ r_2 = 1,\ r_3 = -3$

$\underline{(r+2)(r-1)}(r+3)$

$(r^2 + r - 2)(r+3) = r^3 + 4r^2 + r - 6 = 0$

$\implies \underline{\underline{y''' + 4y'' + y' - 6y = 0}}$

c)
$$\left.\begin{array}{ll} y_1 = y & y_1' = y_2 \\ y_2 = y' & y_2' = y_3 \\ y_3 = y'' & y_3' = 6y_1 - y_2 - 4y_3 \end{array}\right\} \underline{y}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}}_{A} \underline{y}$$

12.5 b) $y'' + y' - 20y = 1$ $\qquad y_e = \underline{\underline{-1/20}}$

Char roots: $r^2 + r - 20 = 0$

$r = \dfrac{-1 \pm \sqrt{1+80}}{2} = \dfrac{-1 \pm 9}{2} = 4, -5$

Unstable

Eq. states: $y' = y'' = 0$

$\left.\begin{array}{l} y = A \\ y' = 0 \quad y'' = 0 \end{array}\right\} -20A = 1 \quad A = -1/20$

Recall:

$$\boxed{\begin{array}{l} y'' + ay' + by = c \qquad y_e = c/b = y_p \\ y = y_h + y_p = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \underline{c/b} \\ y_e = c/b \text{ stable} \iff r_1, r_2 < 0 \end{array}}$$

e) $\underline{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \underline{y}$ or $\underline{y}' = A\underline{y}$

Eq. states: $\underline{y}' = \underline{0} \iff A\underline{y} = \underline{0}$

Stability: $\underline{y_e}$ stable $\iff$ all eigenvalues of A are negative

Eigenvalues: $\begin{vmatrix} -5-\lambda & 0 & 1 \\ 0 & -3-\lambda & 0 \\ 1 & 0 & -5-\lambda \end{vmatrix} = (-3-\lambda)\cdot(\lambda^2+10\lambda+24) = 0$

$(-3-\lambda)\cdot(\lambda+4)(\lambda+6) = 0$

All eq. states are stable $\Longleftarrow$ $\lambda = \underline{-3}$, $\lambda = \underline{-4}$, $\lambda = \underline{-6}$

Eq. states: $A \cdot \underline{y} = \underline{0}$ $\quad |A| = -72 \neq 0$

$\underline{y_e} = \underline{0}$ $\quad$ (A invertible)

② Final exams

Final 01/2024

1 d) $\mathbf{y}' = A\mathbf{y} + \mathbf{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 4 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

Eq. states: $\mathbf{y}' = \mathbf{0} \Leftrightarrow A\mathbf{y} + \mathbf{b} = \mathbf{0}$

$A\mathbf{y} = -\mathbf{b}$

Stability: $\mathbf{y}_e$ stable $\Leftrightarrow$ All eigenval. of A are negative

$\operatorname{tr} A = 1 + (-1) + 4 = 4 > 0 \Rightarrow$ not all eign. val.

$\lambda_1 + \lambda_2 + \lambda_3 = 4$ are negative

Alt: $D_1 = 1 > 0$, $D_2 =$, $D_3 =$ $\Big\}$ not neg. defn.

Concl: no stable eq. state

3 d) $\underbrace{t - 3y}_{p} + \underbrace{(8y - 3t)}_{q} y' = 0$, $y(1) = 0$

$t = 1 \Rightarrow y = 0$

① $t - 3y = h'_t$

② $8y - 3t = h'_y$

① $t - 3y = \cancel{0} - 3y + C'(t)$

② $h = 4y^2 - 3ty + C(t)$

Exact:

$C'(t) = t \quad C(t) = t^2/2$

$\Rightarrow h = 4y^2 - 3ty + t^2/2 = C$

$y(1) = 0: \quad 4 \cdot 0^2 - 3 \cdot 1 \cdot 0 + 1^2/2 = C$

$C = 1/2$

$4y^2 - 3ty + t^2/2 = 1/2 \quad | \cdot 2$

$8y^2 - 6ty + t^2 - 1 = 0$

$y = \dfrac{6t \pm \sqrt{36t^2 - 4 \cdot 8 \cdot (t^2 - 1)}}{2 \cdot 8} = \dfrac{6t}{16} \pm \dfrac{1}{16}\sqrt{4t^2 + 32} = \dfrac{3t}{8} - \dfrac{1}{8}\sqrt{t^2 + 8}$

Final 11/2019, Q4

min $f(\underline{x}) = x^2+y^2+z^2-xy+xz-yz$ when $x+y+z=11$

$= \underline{x}^T A \underline{x}$ $\qquad B\underline{x} = 11$

$$A = \begin{pmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1 & -1/2 \\ 1/2 & -1/2 & 1 \end{pmatrix} \qquad B = (1 \ \ 1 \ \ 1)$$

a) Is f convex? Definiteness of A: pos. defn. Yes, convex

b) $L = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 11)$

FOC: $2A\underline{x} - \lambda \cdot B^T = \underline{0} \Rightarrow 2A\underline{x} = \lambda \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda\\\lambda\\\lambda\end{pmatrix}$

C: $B\underline{x} = 11$

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & \lambda \\ -1 & 2 & -1 & \lambda \\ 1 & -1 & 2 & \lambda \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & \lambda \\ -1 & 2 & -1 & \lambda \\ 2 & -1 & 1 & \lambda \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & \lambda \\ 0 & 1 & 1 & 2\lambda \\ 0 & 1 & -3 & -\lambda \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & \lambda \\ 0 & 1 & 1 & 2\lambda \\ 0 & 0 & -4 & -3\lambda \end{array}\right)$$

$x - y + 2z = \lambda \qquad x = 3\lambda/4$

$y + z = 2\lambda \qquad y = 5\lambda/4$

$-4z = -3\lambda \Rightarrow z = 3\lambda/4$

C: $x+y+z = 11$

$11\lambda/4 = 11$

$\lambda = 4$

Cand. pt: $(x, y, z; \lambda) = (3, 5, 3; 4)$

Is this min? SOC $h = \underline{x}^T A \underline{x} - 4(B\underline{x} - 11)$

$H(h) = 2A$ pos. def. $\Rightarrow h$ convex

$L = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - a)$

$L'_a = \lambda$

$\underset{SOC}{\Rightarrow} f_{min} = f(3,5,3) = \_\_$

c) Env. Thm: min $f = \underline{x}^T A \underline{x}$ when $B\underline{x} = a$ | $a = 11$: $f^*(11) = $

$\lambda^*(a)$ | $\lambda^*(11) = 4$

$f^*(10) = f^*(11) + \Delta a \cdot \frac{df^*(a)}{da} = f(3,5,3) + (-1) \cdot 4$

Final 11/2018, Q2c

$y' = 0{,}15y\,(1 - y/200)$

Eq. states: $y' = 0 \iff \underline{0.15y}\,\underline{(1 - y/200)} = 0$ $\quad$ $y_e = 0$ and $y_e = 200$

$y = 0$ or $y = 200$

Stability:

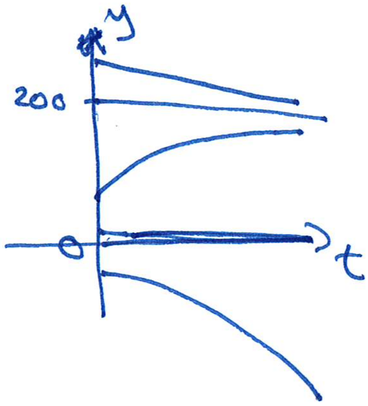
Sign diagram of $F(y) = 0.15y\,(1 - y/200)$

$0{,}15y$ ---0———— (0, 200) y

$1 - y/200$ ————0 - - -

$y' = F(y)$ - 0————0 - - -

← ——→ ←



$y_e = 200$ is stable (not asympt. stable) $\quad$ $y_e = 0$ is unstable

5. Solve $y' = 0{,}15y\,(1 - y/200)$ See Plenary session 3 or solutions.

$y(0) = 50$ → use this to find C

Carrying capacity $= \underline{200}$: $y(t) = 0.90 \cdot 200 = 180$

solve for t.