	Plan	
1	Key Problems:	8.1b, 8.2, 9.1c, 9.2bc, 10.2c, 10.3c, 10.4bc
2	Final exams:	01/2024 Q4e, 01/2022 Q3

① Key problems

8.1 b) max $f = xz + yw$ when $x^2+y^2 \le 1$, $4z^2+9w^2 \le 36$ Std. form

$$\mathcal{L} = xz + yw - \lambda_1(x^2+y^2-1) - \lambda_2(4z^2+9w^2-36)$$

(1) $\mathcal{L}'_x = z - \lambda_1 \cdot 2x = 0$ $\quad x^2+y^2 \le 1$ $\quad \lambda_1 \ge 0$ $\quad \lambda_1 = 0$ if $x^2+y^2<1$

(2) $\mathcal{L}'_y = w - \lambda_1 \cdot 2y = 0$ $\quad 4z^2+9w^2 \le 36$ $\quad \lambda_2 \ge 0$ $\quad \lambda_2 = 0$ if $4z^2+9w^2<36$

(3) $\mathcal{L}'_z = x - \lambda_2 \cdot 8z = 0$

(4) $\mathcal{L}'_w = y - \lambda_2 \cdot 18w = 0$

FOC C CSC

(1)+(3): $z = 2\lambda_1 x$ and (2)+(4): $w = 2\lambda_1 y$

$x - 8\lambda_2(2\lambda_1 x) = 0$ $\quad y - 18\lambda_2(2\lambda_1 y) = 0$

$x(1 - 16\lambda_1\lambda_2) = 0$ $\quad y(1 - 36\lambda_1\lambda_2) = 0$

$x=0$ or $\lambda_1\lambda_2 = 1/16$ $\quad y=0$ or $\lambda_1\lambda_2 = 1/36$

$\Rightarrow$ (a) $x=0, y=0$ (b) $x=0, \lambda_1\lambda_2 = \frac{1}{36}$ (c) $\lambda_1\lambda_2 = \frac{1}{16}, y=0$

(a) $x=0 \Rightarrow z=0$, $y=0$, $w=0$; $x^2+y^2<1$, $4z^2+9w^2<36$; $\lambda_1=0$, $\lambda_2=0$ $\quad (0,0,0,0;0,0)$ $f=0$

(b) $x=0 \Rightarrow z=0$; $\lambda_1>0, \lambda_2>0 \Rightarrow x^2+y^2=1$, $4z^2+9w^2=36$; $y=\pm 1$, $w=\pm 2$; (2) $\lambda_1 = \frac{w}{2y} = \pm 1$ $\Rightarrow (0,1,0,2;1,1/36)$, $(0,-1,0,-2;1,1/36)$ $\quad f=2$

(c) $y=0 \Rightarrow w=0$; $\lambda_1>0, \lambda_2>0 \Rightarrow x^2+y^2=1$, $4z^2+9w^2=36$; $x=\pm 1$, $z=\pm 3$; (1) $\lambda_1 = \frac{z}{2x} = \pm 3/2$ $\Rightarrow (1,0,3,0;3/2,1/24)$, $(-1,0,-3,0;3/2,1/24)$ $\quad f=3$

$\lambda_2 = \frac{1}{16}\cdot\frac{1}{\lambda_1} = \frac{1}{16}\cdot\frac{2}{3}$

Note: $x^2+y^2 \le 1$ $x^2, y^2 \le 1$ $-1 \le x, y \le 1$
$4z^2+9w^2 \le 36$ $z^2 \le 9, w^2 \le 4$ $-3 \le z \le 3, -2 \le w \le 2$ } bounded set $\Downarrow$ compact set

EVT: there is a max $\Rightarrow$ max is $f_{max} = 3$
or
~~a pt where NDCQ fails.~~

NDCQ: $J = \begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 8z & 18w \end{pmatrix}$

BB: $\mathrm{rk}\, J = 2$ $\quad \mathrm{rk}\, J < 2$ if $x=y=0$ or $z=w=0$
NB NB.

BN: $\mathrm{rk}\,(2x \;\; 2y \;\; 0 \;\; 0) = 1$ $\quad \mathrm{rk} < 1$ $\quad x=y=0$
NB

NB: $\mathrm{rk}\,(0 \;\; 0 \;\; 8z \;\; 18w) = 1$ $\quad \mathrm{rk} < 1$ $\quad z=w=0$
NB

no adm pts where NDCQ fails $\Rightarrow$

$\Rightarrow$ $f_{max} = \underline{\underline{3}}$ at $(1,0,3,0), (-1,0,-3,0)$ with $\lambda_1 = 3/2$ $\lambda_2 = 1/24$

8.2 a) max $f = x^2y^2z^2$ when $x^2+y^2+z^2+x^2y^2z^2 \le 4$ std. form

$\mathcal{L} = x^2y^2z^2 - \lambda(x^2+y^2+z^2+x^2y^2z^2-4)$

$x^2 \le 4, y^2 \le 4, z^2 \le 4$
$-2 \le x, y, z \le 2$ $\Rightarrow$ compact set
$\xrightarrow{EVT}$ there is a max.

(1)
$\mathcal{L}'_x = 2xy^2z^2 - \lambda \cdot (2x + 2xy^2z^2) = 0$
$\mathcal{L}'_y = 2yx^2z^2 - \lambda\,(2y + 2yx^2z^2) = 0$
$\mathcal{L}'_z = 2zx^2y^2 - \lambda\,(2z + 2zx^2y^2) = 0$
FOC

$x^2+y^2+z^2+x^2y^2z^2 \le 4$
C

$\lambda \ge 0$
$\lambda = 0$ if B
CSC

(1) $2x\,(y^2z^2 - \lambda \cdot (1+y^2z^2)) = 0$
(2) $2y\,(x^2z^2 - \lambda\,(1+x^2z^2)) = 0$
(3) $2z\,(x^2y^2 - \lambda \cdot (1+x^2y^2)) = 0$

Probably a bunch of cand. pts with $x=0$ or $y=0$ or $z=0$ with $f=0$
$\to$ probably not max pts.

Assume $x,y,z \neq 0$:

(1) $y^2z^2 - \lambda \cdot (1+y^2z^2) = 0 \Rightarrow \lambda = \frac{y^2z^2}{1+y^2z^2} \neq 0 \Rightarrow$ B constraint

(2) $x^2z^2 - \lambda(1+x^2z^2) = 0 \Rightarrow \lambda = \frac{x^2z^2}{1+x^2z^2}$

(3) $x^2y^2 - \lambda(1+x^2y^2) = 0$

$\lambda = \frac{x^2y^2}{1+x^2y^2}$

$$\frac{y^2z^2}{1+y^2z^2} = \frac{x^2z^2}{1+x^2z^2} \quad | \cdot \text{common denom.}$$

$$\frac{x^2y^2}{1+x^2y^2} = \frac{x^2z^2}{1+x^2z^2} \quad | \cdot \text{common denom.}$$

$y^2z^2(1+\cancel{x^2z^2}) = x^2z^2(1+\cancel{y^2z^2})$

$y^2z^2 = x^2z^2 \quad | : z^2$

$\underline{y^2 = x^2}$

$x^2y^2(1+\cancel{x^2z^2}) = x^2z^2(1+\cancel{x^2y^2})$

$x^2y^2 = x^2z^2 \quad | : x^2$

$\underline{y^2 = z^2}$

$\Rightarrow \underline{x^2 = y^2 = z^2} = a$

C: $x^2+y^2+z^2+x^2y^2z^2 = 4$

$3a + a^3 = 4$

$a^3 + 3a - 4 = 0$

$(a-1)\cdot(a^2+a+4) = 0$

polyn. div:
$a^3+3a-4 : a-1 = a^2+a+4$

$a = 1$ or $\cancel{a = \frac{-1 \pm \sqrt{1-4\cdot 4}}{2}}$

$\Rightarrow \underline{x^2 = y^2 = z^2 = 1}$

Cand pts: $(\pm 1, \pm 1, \pm 1; 1/2)$ $\quad f = 1 \quad \leftarrow f_{max} = 1$

$x^2+y^2+z^2+x^2y^2z^2 \leq 4$

NDCQ: $J = \begin{pmatrix} 2x + 2xy^2z^2 & 2y + 2yx^2z^2 & 2z + 2zx^2y^2 \end{pmatrix}$

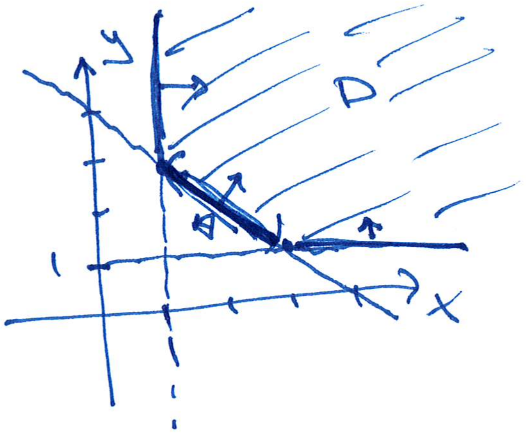
$\qquad = 2x(1+y^2z^2) \quad 2y(1+x^2z^2) \quad 2z(1+x^2y^2)$

rk $J = 1$: B, NDCQ fails: rk $< 1 \Rightarrow x = y = z = 0$ NR

No adm pts where NDCQ fails

8.2b) max $f = \ln(x^2 y) - x - y = 2\ln x + \ln y - x - y = 2\ln x - x + \ln y - y$ when

$x \geq 1$, $y \geq 1$, $x + y \geq 4$ $\xrightarrow{\text{std. form}}$ $-x \leq -1$, $-y \leq -1$, $-x - y \leq -4$



D closed, <u>not</u> bounded

$$\mathcal{L} = 2\ln x + \ln y - x - y - \lambda_1(-x+1) - \lambda_2(-y+1) - \lambda_3(-x-y+4)$$

(A) $x+y=4$ "NNB"
$x>1$
$y>1$

FOC: $\mathcal{L}'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_3 = 0$, $\mathcal{L}'_y = \frac{1}{y} - 1 + \lambda_2 + \lambda_3 = 0$

C: $x \geq 1$, $y \geq 1$, $x + y \geq 4$

CSC: $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$
$\lambda_1 = 0$ if $x > 1$
$\lambda_2 = 0$ if $y > 1$
$\lambda_3 = 0$ if $x + y > 4$

KT cond. for all cases.

(A) $x>1$ $\lambda_1 = 0$; $y > 1$ $\lambda_2 = 0$; $x+y=4$ $\lambda_3 \geq 0$

FOC: $\frac{2}{x} - 1 + \lambda_3 = 0$ $\quad \lambda_3 = 1 - 2/x$

$\frac{1}{y} - 1 + \lambda_3 = 0$ $\quad \lambda_3 = 1 - 1/y$

$\Big\}$ $2/x = 1/y$ $\quad 2y = x$

$x + y = 4$

$2y + y = 4$ $\quad 3y = 4$ $\quad y = \underline{4/3}$ $\quad x = \underline{8/3}$ $\quad \lambda_3 = 1 - 2 \cdot 3/8 = 2/8 = \underline{1/4}$

$(x, y; \lambda_1, \lambda_2, \lambda_3) = (8/3, 4/3; 0, 0, 1/4)$ ok cand. pt (check ineq.)! $f = f(8/3, 4/3)$

<u>SOC:</u> $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1/4$

$$h(x,y) = h(x,y;0,0,1/4)$$
$$= 2\ln x + \ln y - x - y - \overset{1/4}{\overset{\shortparallel}{\lambda_3}}(-x-y+4)$$
$$= 2\ln x + \ln y - x - y - \tfrac{1}{4}(-x-y+4)$$

$$h'_x = 2/x + \ldots = 2x^{-1} + \ldots$$
$$h'_y = 1/y + \ldots = y^{-1} + \ldots$$

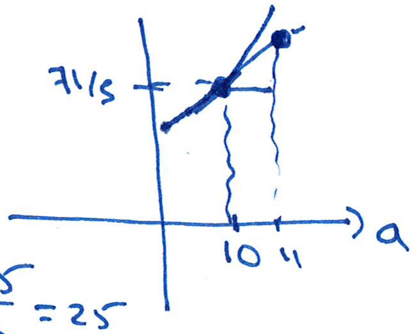
$$H(h) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$$

neg. def.
$\Downarrow$
h concave
$\Downarrow$ SOC

$(x,y) = \underline{\underline{(8/3, 4/3)}}$ is max pt.

<u>9.1</u> c) <u>max-pb with a parameter:</u>

$$\max \underbrace{16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + az}_{f}$$



<u>a=10:</u> a) b)

<u>a=11:</u> $f^*(11) \simeq \underset{71/3}{\underset{\shortparallel}{f^*(10)}} + \underset{+1}{\underset{\shortparallel}{\Delta a}} \cdot \underset{4/3}{\underset{\shortparallel}{\dfrac{df^*(a)}{da}}} = \dfrac{75}{3} = \underline{\underline{25}}$

$f'_a = z \implies$ Envth. $\dfrac{df^*(a)}{da} = f'_a(x^*(a), y^*(a), z^*(a))$
$$= \underset{\underset{a=10}{\uparrow}}{z^*(a)} = 4/3$$

<u>Solved in a):</u>
a=36 b=18 c=−72

9.2 bc) $f = 4x^2 + \ldots + ax + by + cz + 200$

<u>Env:</u> $\left.\begin{array}{l} f'_a = x \\ f'_b = y \\ f'_c = z \end{array}\right\}$ $f^*(a,b,c) \simeq f^*(36,18,-72) + \Delta a \cdot \overbrace{\dfrac{df^*(a)}{da}}^{=x^*(a)}$
$$+ \Delta b \cdot \underbrace{\dfrac{df^*(b)}{db}}_{y^*(b)} + \Delta c \cdot \dfrac{df^*(c)}{dc} = z^*(c)$$

10.2c) $y' = 5y(1-y/10) = 5 \cdot \underline{y \cdot (1-y/10)}$ Sep.

$$10 \cdot \frac{1}{10 \cdot y(1-y/10)} y' = 5$$

$$\frac{10}{y(10-y)} = \overset{=1}{\frac{A}{y}} + \overset{=1}{\frac{B}{10-y}} \quad | \cdot \text{cd.}$$

$$10 = A \cdot (10-y) + By$$

$y=0$: $10 = A \cdot 10 + \cancel{B \cdot 0}$ $\quad \underline{A=1}$

$y=10$: $10 = \cancel{A \cdot 0} + B \cdot 10$ $\quad \underline{B=1}$

$$\int \frac{10}{y(10-y)} \underbrace{y'\,dt}_{dy} = \int 5\,dt$$

$$\int \frac{1}{y} + \frac{1}{10-y}\,dy = 5t + C$$

$$\ln|y| - \ln|10-y| = 5t + C \quad | e^{\cdot}$$

$$e^{\ln|y| - \ln|10-y|} = e^{5t+C}$$

$$\frac{|y|}{|10-y|} = e^{C} \cdot e^{5t}$$

$$\frac{y}{10-y} = \pm e^{C} \cdot e^{5t} = K \cdot e^{5t} \quad | \cdot (10-y)$$

$$y = Ke^{5t} \cdot (10-y)$$

$$y + Ke^{5t} \cdot y = 10Ke^{5t}$$

$$\frac{y(1+Ke^{5t})}{1+Ke^{5t}} = \frac{10Ke^{5t}}{1+Ke^{5t}}$$

$$\underline{\underline{y = \frac{10K \cdot e^{5t}}{1+Ke^{5t}}}}$$

10.3 c) $y' + 2y = e^t$ $\underline{\text{lin.}}$ $\underline{a(t) = 2}$ $b(t) = e^t$

$(y' + 2y)e^{2t} = e^t \cdot e^{2t}$ $\underline{\text{Int. factor:}}$ $\int 2\,dt = 2t + C$

$(y \cdot e^{2t})' = e^{3t}$ $\qquad u = \underline{e^{2t}}$

$$y \cdot e^{2t} = \int e^{3t}\,dt = \int e^u \frac{du}{3} = \frac{1}{3}e^u + C$$

$u = 3t$

$du = 3dt$

$$\frac{y \cdot e^{2t}}{e^{2t}} = \frac{\frac{1}{3}e^{3t} + C}{e^{2t}} = \frac{1}{3}e^t + \frac{C}{e^{2t}}$$

$$\underline{\underline{y = \frac{1}{3}e^t + Ce^{-2t}}}$$

$\underline{10.4}$ b) $\underbrace{2y - 3t^2}_{p} + \underbrace{2(y+t)}_{q}\, y' = 0$ $\qquad \boxed{p + q \cdot y' = 0}$

① $h'_t = 2y - 3t^2$ $\qquad$ ① $h = \underline{2yt - t^3 + Q(y)} = 2yt - t^3 + y^2$

② $h'_y = 2y + 2t$ $\qquad \Rightarrow$ ②: $\cancel{2t} \cancel{-0} + Q'(y) = 2y + \cancel{2t}$

$Q'(y) = 2y$

$Q(y) = y^2$

$\Rightarrow$ The eqn. $\underline{\text{is exact}}$ and

$h(t,y) = C$

$2yt - t^3 + y^2 = C$ $\qquad$ quadr. form. $\rightarrow$ $y = \dfrac{-2t \pm \sqrt{(2t)^2 - 4 \cdot 1 \cdot (-t^3 - C)}}{2 \cdot 1}$

$y^2 + 2t \cdot y + (-t^3 - C) = 0 \Rightarrow$ $\qquad \underline{y = -t \pm \frac{1}{2}\sqrt{4t^2 + 4t^3 + 4C}}$

Final 01/2024, Q4e

$$\min f = 2xz + 2xw + 2yz + 2yw \quad \text{when} \quad x^2 + 2y^2 + 2z^2 + 6w^2 = 48$$

$$= \underline{x}^T A_1 \underline{x} \qquad\qquad \underline{x}^T A_2 \underline{x} = 48$$

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$L = \underline{x}^T A_1 \underline{x} - \lambda(\underline{x}^T A_2 \underline{x} - 48)$$

FOC: $L'(\underline{x}) = 2A_1\underline{x} - \lambda \cdot 2A_2\underline{x} = \underline{0} \quad |:2$ $\qquad$ C: $\underline{x}^T A_2 \underline{x} = 48$

$$A_1\underline{x} - \lambda A_2 \underline{x} = \underline{0}$$

$$(A_1 - \lambda A_2)\underline{x} = \underline{0}$$ ← 4×4 hom. lin. sys with par. λ

$$|A_1 - \lambda A_2| = 0 \text{ or } \cancel{\underline{x} = \underline{0}}$$

$$\begin{array}{c} \\ \frac{1}{2}\cdot \\ \frac{1}{2}\cdot \\ \frac{1}{6}\cdot \end{array} \begin{vmatrix} -\lambda & 0 & 1 & 1 \\ 0 & -2\lambda & 1 & 1 \\ 1 & 1 & -2\lambda & 0 \\ 1 & 1 & 0 & -6\lambda \end{vmatrix} = 0 \rightsquigarrow \underbrace{\begin{vmatrix} -\lambda & 0 & 1 & 1 \\ 0 & -\lambda & 1/2 & 1/2 \\ 1/2 & 1/2 & -\lambda & 0 \\ 1/6 & 1/6 & 0 & -\lambda \end{vmatrix}}_{\text{Eigenvalues of } P} = 0 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} = 0$$

$$P = \begin{vmatrix} 0 & 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{1/2} & \cancel{1/2} \\ 1/2 & 1/2 & 0 & 0 \\ \cancel{1/6} & \cancel{1/6} & \cancel{0} & \cancel{0} \end{vmatrix}$$

We know: $\lambda_1 = 1$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$

rk $P = 2$: $\lambda_2 = \lambda_3 = 0$

$\lambda_4 = -1$

rk $P = 2 \Rightarrow P\underline{x} = \underline{0}$ has 2 free var.

Idea: Find all crit. pts with $\lambda = -1$
Check if they are min using SOC.

$$h = \underline{x}^T A_1 \underline{x} + 1 \cdot (\underline{x}^T A_2 \underline{x} - 48)$$

$$H(h) = 2A_1 + 2A_2 = 2(A_1 + A_2)$$

$$2\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 6 \end{pmatrix} \leftarrow \begin{array}{l} D_1 = 1 \\ D_2 = 2 \\ D_3 = 3 + (-2) = 1 \\ D_4 = \end{array}$$