	Plan	
1	Key Problems:	5.1cde, 5.2d, 5.3, 5.4bc, 5.5bf, 6.2b, 6.3cd, 6.4bce, 6.5ab, 7.1d, 7.2b, 7.3b, 7.4d
2	Mock Midterm Exam:	10/2024 Q7, Q8

5.1 c) $D_1 = 4$ pos. defn.
$D_2 = 20$
$D_3 = 5(16-1) > 0$

d) $D_1 = 2$ $rk = 2$
$D_2 = 14 - 9 = 5$ $\Downarrow$ RRC
$D_3 = -5(35) + 35 \cdot 5 = 0$ pos. semidefn.

e) $D_1 = -1$ $rk = 2$ RRC does not work
$D_2 = 0$
$D_3 = 0$ $R(2) = 2 \cdot R(1)$ $\Delta_2^{23} = \begin{vmatrix} -4 & -4 \\ -4 & -2 \end{vmatrix} = 8 - 16$
$= -8 < 0$
$\Delta_1 = -1, -4, -2$
$\Delta_2 = 0, -8, -2 \longrightarrow$ indefn.

$\boxed{\text{Remember: } \Delta_2, \Delta_4, \ldots \text{ (even order) negative} \Rightarrow \text{indefn.}}$

5.2 d)
$$A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$
$\Delta_2^{23} = 0 - \frac{1}{4} = -1/4 < 0$
Indefn.

5.3

$D_1 = 1$

$D_2 = 1$

$D_3 = 1 \cdot (1-1) = 0$

$\left.\begin{array}{l} R(3) = -R(2) \\ R(4) = -R(1) \end{array}\right\}$ $rk = 2$

$\Downarrow$ free

pos. semidef.

5.4

b) $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = (4-\lambda)\cdot(\lambda^2 - 8\lambda + 15) = 0$

$\lambda = 4$ or $\lambda = 3, \lambda = 5$

$\lambda_1 = 3$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \underline{v_1} \to \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$

$\lambda_2 = 4$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{v_2} \to \frac{1}{1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $P = \begin{pmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$

$\lambda_3 = 5$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underline{v_3} \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{u} = P^T \cdot \underline{x}$

$\Downarrow$

$\underline{v_1} \cdot \underline{v_2} = 0 \quad \underline{v_1} \cdot \underline{v_3} = 0 \quad \underline{v_2} \cdot \underline{v_3} = 0$

$\underline{\underline{3u^2 + 4v^2 + 5w^2}}$

c) $\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\lambda = -1$ is one solution of mult. 2

$\lambda_1 = \lambda_2 = -1, \ \lambda_3 = 2$ since

$\lambda_1 + \lambda_2 + \lambda_3 = 0$

$-2 + \lambda_3 = 0$

$\lambda_3 = 2$

E_1: $\begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{1} \end{pmatrix}$ $x = -y - z$

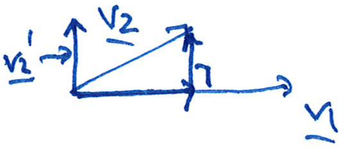
y, z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y \cdot \overbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}^{=v_1} + z \cdot \overbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}^{=v_2}$

$\underline{E_2}$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ $\quad \underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

$\underline{v_1} \cdot \underline{v_3} = 0, \; \underline{v_2} \cdot \underline{v_3} = 0, \; \underline{v_1} \cdot \underline{v_2} = 1 \neq 0$



$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{v_1 \cdot v_2}{v_1 \cdot v_1} \cdot \underline{v_1}$

$$\underline{\underline{-u^2 - v^2 + 2w^2}}$$

$$\underline{v_2'} = \underline{v_2} - \text{proj}_{\underline{v_1}}(\underline{v_2})$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v_2'} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\perp \qquad\qquad \perp \qquad\qquad \perp$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\parallel$$

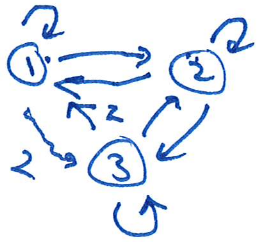
$$\frac{1}{\sqrt{3/2}}\begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}$$

$\underline{5.5\ b)}$ $\underline{\text{regular}}$ since $\underline{A > 0}$ $\begin{pmatrix} a_{11} > 0 & a_{12} > 0 \\ a_{21} > 0 & a_{22} > 0 \end{pmatrix}$

$\underline{\lambda = 1}$: $\begin{pmatrix} -0.23 & 0.46 \\ \cancel{0.23} & \cancel{-0.46} \end{pmatrix}$ $\quad -0.23x + 0.46y = 0$, y free $\quad x = \frac{-0.46y}{-0.23} = 2y$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rightarrow \underline{v} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}}}$$

f) Graph:



$A^2 > 0$

regular

Remember:

Markov chain is regular $\Leftrightarrow$ $A^N > 0$ for some N

6.2 b)

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \quad \left.\begin{array}{l} D_1 = -1 \\ D_2 = 1 \\ \text{rk } A = 2 \end{array}\right\}$$

RRE $\Rightarrow$ A neg semidefin.

$H(f) = 2A$ –"–

$\Rightarrow f$ concave

$R(3) = -R(1)$
$R(4) = R(2)$

$f(x) = x^T A x$
$f'(x) = 2Ax$
$f_{max} = f(0,0,0,0) = \underline{\underline{0}}$ (no min)

6.3 d)

$f'_x = -4x^3 - 4x + 6z = 0$
$f'_y = -6y = 0$
$f'_z = 6x - 12z = 0$

$y = 0$

$z = \frac{6x}{12} = \frac{x}{2}$

$-4x^3 - 4x + 6(x/2) = 0$
$-4x^3 - x = 0$
$-x(4x^2+1) = 0$
$x = 0 \quad 4x^2+1 > 0$

Stat. pts: $(x,y,z) = (0,0,0)$

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$D_1 = -12x^2 - 4 < 0$
$D_2 = -6D_1 > 0$
$D_3 = -6 \cdot (144x^2 + 48 - 36)$
$= -6(144x^2 + 12) < 0$

neg. defin. for (x,y,z)
$\Rightarrow f$ concave
$\Rightarrow f_{max} = f(0,0,0) = \underline{\underline{16}}$
global (and local) max

6.4 b) $f = e^u$, $u = x - 2y + z$

$f'_x = e^u \cdot u'_x = e^u \cdot 1$

$f'_y = \quad = e^u \cdot (-2)$

$f'_z = \quad = e^u \cdot (1)$

$$H(f) = \begin{pmatrix} e^u \cdot 1 \cdot 1 & e^u \cdot 1 \cdot (-2) & e^u \cdot 1 \cdot 1 \\ e^u \cdot (-2) \cdot 1 & e^u \cdot (-2) \cdot (-2) & e^u \cdot (-2) \cdot 1 \\ e^u \cdot 1 \cdot 1 & e^u \cdot 1 \cdot (-2) & e^u \cdot 1 \cdot 1 \end{pmatrix}$$

$$= \underset{\text{pos.}}{e^u} \cdot \underbrace{\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}}_{\substack{D_1 = 1 \\ rk = 1}} \underset{\text{RRC}}{\Rightarrow} \text{pos. semidef.}$$

f convex

e) $f = \dfrac{xy + xz + yz}{xyz}$, $x, y, z > 0$

$$= \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} = \frac{1}{z} + \frac{1}{y} + \frac{1}{x}$$

$$= x^{-1} + y^{-1} + z^{-1}$$

$f'_x = -1 \cdot x^{-2}$

$f'_y = -1 \cdot y^{-2}$

$f'_z = -1 \cdot z^{-2}$

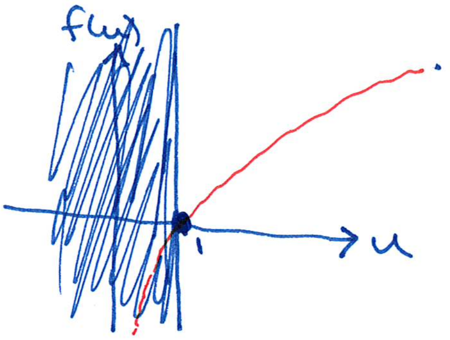
$$H(f) = \begin{pmatrix} 2x^{-3} & 0 & 0 \\ 0 & 2y^{-3} & 0 \\ 0 & 0 & 2z^{-3} \end{pmatrix} = \begin{pmatrix} 2/x^3 & 0 & 0 \\ 0 & 2/y^3 & 0 \\ 0 & 0 & 2/z^3 \end{pmatrix}$$

$\lambda_1 = 2/x^3$, $\lambda_2 = 2/y^3$, $\lambda_3 = 2/z^3 > 0$

f convex

6.5 a) $f(x,y,z) = \ln(1 + \underline{2x^2+2xy+3y^2-2xz+z^2})$

$= \ln(u)$, $u = 1 + \underline{x}^T A \underline{x}$ $\quad A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix}$



$u_{min} = u(0,0,0)$

$= 1$

$V_u = [1, \rightarrow)$

$D_1 = 2$

$D_2 = 5$

$D_3 = -1(3) + 1 \cdot 5 = 2$

pos. defn.

u convex

$f(u) = \ln(u)$, $u \geq 1$

$f(1) = \ln(1) = 0$

$f'(u) = \frac{1}{u} > 0$

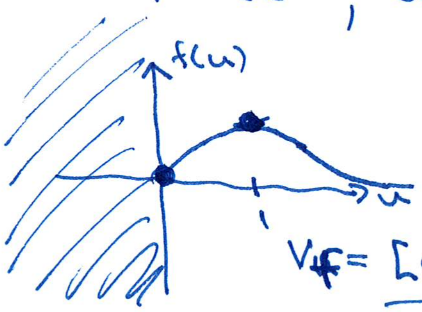
$V_f = \underline{\underline{[0, \rightarrow)}}$

b) $f(x,y,z) = (x^2+y^2+z^2)e^{-x^2-y^2-z^2}$

$= u e^{-u}$, $u = x^2+y^2+z^2$

$= \underline{x}^T A \underline{x}$, $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$f(u) = ue^{-u}$, $u \geq 0$



$V_f = \underline{\underline{[0, 1/e]}}$

$u_{min} = u(0,0,0)$

$= 0$

$\underline{V_u = [0, \rightarrow)}$

pos. defn.

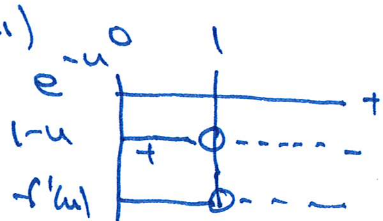
$\Rightarrow$ u convex

$f(0) = 0 \cdot e^{-0} = 0$ $\quad f'(u) = 1 \cdot e^{-u} + u \cdot e^{-u}(-1)$

$f(1) = 1 \cdot e^{-1} = 1/e$ $\quad$ "$\frac{\infty}{\infty}$" $\quad = e^{-u} \cdot (1-u)$

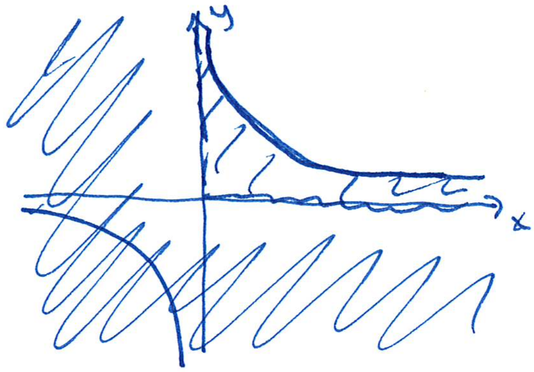
$f(u) = ue^{-u} = \frac{u}{e^u} \rightarrow 0$

when $u \to \infty$



7.1 d) D: $4xy \le 1$

$x, y > 0$

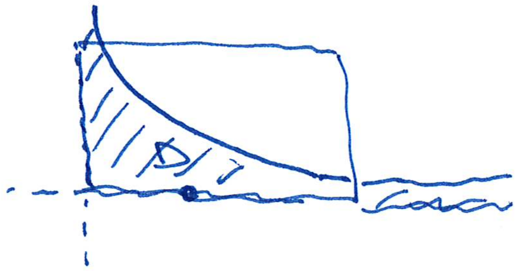


$4xy = 1$

$y = \frac{1}{4x} = \frac{1}{4} \cdot \frac{1}{x}$

$4xy < 1$

$y < \frac{1}{4x}$



not compact

not closed

not bounded

7.4 d)

$$\left.\begin{aligned} xy - zw &= 1 \\ x + y + z + w &= 4 \end{aligned}\right\} C$$

NDCQ: $\text{rk } J = 2$

NDCQ fails: $\text{rk } J < 2$

~~$x = y = z = w = 0$~~

~~$(x, y, z, w) = (0,0,0,0)$ is not adm.~~

$$J = \begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ y & x & -w & -z \end{pmatrix} \xrightarrow{-y} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x-y & -w-y & -z-y \end{pmatrix}$$

Remember: Exceptional candidate pts = $\boxed{\text{rk } J < m} + C$

No adm. pts such that NDCQ fails.

$(x, x, -x, -x)$ is not adm.

$0 \neq 4$

$$\text{rk} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & x-y & -w-y & -z-y \end{pmatrix} < 2$$

$x - y = 0$ $\quad -w - y = 0$ $\quad -z - y = 0$

$y = x$ $\quad w = -y = -x$ $\quad z = -y = -x$

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7.3b) max/min $f = xw - yz$ when $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$

Method of Lagrange multipliers:

$$L(x,y,z,w;\lambda_1,\lambda_2) = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

FOC:
$$\boxed{\begin{aligned} L'_x &= w - \lambda_1(2x) = 0 && (1)\\ L'_y &= -z - \lambda_1(8y) = 0 && (2)\\ L'_z &= -y - \lambda_2(8z) = 0 && (3)\\ L'_w &= x - \lambda_2(18w) = 0 && (4) \end{aligned}}$$

C:
$$\boxed{\begin{aligned} x^2+4y^2 &= 4 && (5)\\ 4z^2+9w^2 &= 36 && (6) \end{aligned}}$$

(1)+(4): (1) $w = 2\lambda_1 x$

(4) $x - 18\lambda_2(2\lambda_1 x) = 0$

$x \cdot (1 - 36\lambda_1\lambda_2) = 0$

$\boxed{x=0}$ or $\boxed{\lambda_1\lambda_2 = 1/36}$

(2)+(3): $z = -8\lambda_1 y$

$-y - 8\lambda_2(-8\lambda_1 y) = 0$

$-y(1 - 64\lambda_1\lambda_2) = 0$

$\boxed{y=0}$ or $\boxed{\lambda_1\lambda_2 = 1/64}$

(a) $x=0, y=0$: $w=0, z=0$, $x^2+4y^2=0 \neq 4$ no points

(b) $x=0$, $\lambda_1\lambda_2 = 1/64$: $w=0$ $\left\{\begin{matrix} 4y^2=4 & y^2=1 & y=\pm 1 \\ 4z^2=36 & z^2=9 & z=\pm 3 \end{matrix}\right\} \Rightarrow (0,\pm 1,\pm 3,0)$

(c) $\lambda_1\lambda_2 = 1/36$, $y=0$

(2) $z = -8\lambda_1 y$

$\lambda_1 = \dfrac{z}{-8y}$

$(0,1,3,0)$: $\lambda_1 = -3/8$ $\lambda_2 = -\frac{1}{24}$

$(0,1,-3,0)$: $\lambda_1 = 3/8$ $\lambda_2 = 1/24$

$(0,-1,3,0)$: $\lambda_1 = 3/8$ $\lambda_2 = 1/24$

$(0,-1,-3,0)$: $\lambda_1 = -3/8$ $\lambda_2 = -1/24$

$(0,1,3,0; -3/8, -1/24)$ $f=-3$

$(0,1,-3,0; 3/8, 1/24)$ $f=3$

$(0,-1,3,0; 3/8, 1/24)$ $f=3$

$(0,-1,-3,0; -3/8, -1/24)$ $f=-3$

(c) $\lambda_1\lambda_2 = 1/36,\ y=0$: $z=0$ $\quad x^2=4 \quad x=\pm 2$ $\quad 9w^2=36 \quad w=\pm 2$

$(2,0,0,2;\ 1/2,\ 1/18)$ $f=4$

$(2,0,0,-2;\ -1/2,\ -1/18)$ $f=-4$

$(-2,0,0,2;\ -1/2,\ -1/18)$ $f=-4$

$(-2,0,0,-2;\ 1/2,\ 1/18)$ $f=4$

(1) $w - 2\lambda_1 x = 0$

$\lambda_1 = \dfrac{w}{2x}$

Best cand. for max: $(2,0,0,2)$ $\lambda_1 = 1/2$ $\lambda_2 = 1/18$

$(-2,0,0,-2)$ — " —

SOC: $h(x,y,z,w) = L(x,y,z,w;\ 1/2,\ 1/18) = xw - yz - \frac{1}{2}(x^2+4y^2-4)$

$- \frac{1}{18}(4z^2+9w^2-36)$

$= \underline{x}^T A \underline{x} + 2 + 2$

$$A = \begin{pmatrix} -1/2 & 0 & 0 & 1/2 \\ 0 & -2 & -1/2 & 0 \\ 0 & -1/2 & -2/9 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}$$

$D_1 = -1/2 < 0$

$D_2 = 1 > 0$

$D_3 = -1/2 \cdot (4/9 - 1/4)$

$= -\frac{1}{2}\left(\frac{16-9}{36}\right) < 0$

$D_4 = 0$ $[R(4) = -R(1)]$

A negative semidef. $\xLeftarrow{\text{RRC}}$ rk $A = 3$

$\Updownarrow$

h concave $\xRightarrow{\text{SOC}}$ $f_{max} = 4$ at $(2,0,0,2)$, $(-2,0,0,-2)$ with $\lambda_1 = 1/2$, $\lambda_2 = 1/18$

To show that $f_{min} = -4$ at $(2,0,0,-2)$, $(-2,0,0,2)$, with $\lambda_1 = -1/2$, $\lambda_2 = -1/18$ we can use SOC on these cand. pts in a similar way