Plan

1 Envelope Theorems
2 Minimum variance portfolio problems

① Envelope theorems

Ex: max $f(x,y,z,w) = xw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le 36 \end{cases}$ KT pb. in std. form

Solution: $f_{max} = \underline{\underline{4}}$ at $(x,y,z,w) = \begin{cases} (2,0,0,2), \\ (-2,0,0,-2) \end{cases}$ with $\begin{cases} \lambda_1 = 1/2 \\ \lambda_2 = 1/18 \end{cases}$

$f^*(36)$; $\underline{x}^*(36)$; $\underline{\lambda}^*(36)$

What about: max $f = xw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le \underline{\underline{35}} \end{cases}$

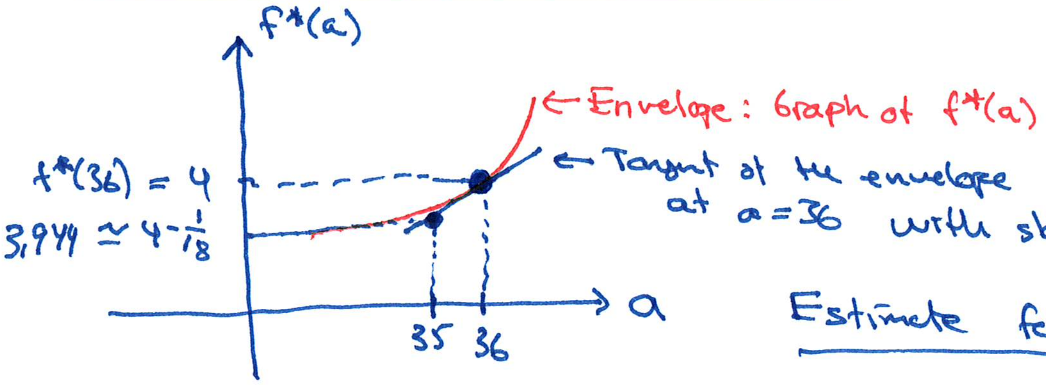
Idea: Consider max $f = xw - yz$ when $\begin{cases} x^2+4y^2 \le 4 \\ 4z^2+9w^2 \le \mathbf{a} \end{cases}$ KT pb with par. $\underline{a}$

Notation: $f^*(a)$ = maximum value with parameter a
= optimal value function

$\left.\begin{matrix} x^*(a) \\ y^*(a) \\ z^*(a) \\ w^*(a) \end{matrix}\right\}$ = maximum point — " —

$\left.\begin{matrix} \lambda_1^*(a) \\ \lambda_2^*(a) \end{matrix}\right\}$ = Lagrange multipliers — " —

Question: $f^*(36) = 4$ $f^*(35) = ?$



Estimate for $f^*(35)$:

$$f^*(35) \simeq f^*(36) + \Delta a \cdot \frac{df^*(a)}{da}$$

$f^*(36) = 4$, $\Delta a = 35-36 = -1$, $\frac{df^*(a)}{da} = \,? = 1/18$

$$= 4 - 1/18 \simeq \underline{\underline{3.944}}$$

Envelope Thm:

$$\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a);\underline{\lambda}^*(a))$$

In this case:

$$\mathcal{L} = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-a) = \dots + \lambda_2 a$$

$$L'_a = \lambda_2 \Rightarrow L'_a(\underline{x}^*(a);\underline{\lambda}^*(a)) = \lambda_2^*(a) \underset{a=36}{=} \lambda_2^*(36) = \frac{1}{18}$$

Interpretation of Lagrange multipliers

λ_i = marginal change in optimal value per unit change in a_i, the constant in the corresponding constraint $\boxed{g_i(\underline{x}) \le a_i}$

In this case

max $xw-yz$ wh. $x^2+4y^2 \le 4$, $4z^2+9w^2 \le \underline{\underline{36}}$

$\lambda_1 = \frac{1}{2}$

$\lambda_2 = \frac{1}{18}$

Ex: max $f = \underset{b}{2}xw - yz$ when $x^2+4y^2 \le 4$, $4z^2+9w^2 \le 36$

$$L = bxw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

$$L'_b = xw \underset{\text{Env. Thm}}{\Rightarrow} \frac{df^*(b)}{db} = x^*(b)\,w^*(b) = x^*(1)\,w^*(1) = 2\cdot 2 = (-2)(-2) = 4$$

$b=1$

$b=1$: $f^*(1) = 4$

$x^*(1) = 2$ or -2

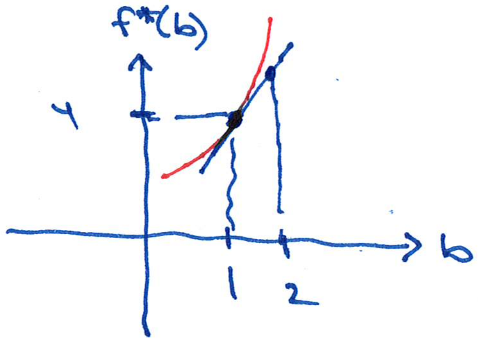
$y^*(1) = 0$

$z^*(1) = 0$

$w^*(1) = 2$ or -2

$\lambda_1^*(1) = 1/2$

$\lambda_2^*(1) = 4/18$



$$f^*(2) \simeq \underset{4}{f^*(1)} + \underset{2-1=1}{\Delta b} \cdot \underset{4}{\frac{df^*(b)}{db}}$$

$= 8$

Ex: $\max \; f(x) = 1 + 2x - x^2$

$f' = 2 - 2x = 0$
$\underline{x = 1}$

$f'' = -2 < 0$ f concave ∩

$f_{max} = f(1) = \underline{\underline{2}}$
at $\underline{x = 1}$

What about: $\max \; f = 1 + ax - x^2$

$f'_x = a - 2x = 0$
$\underline{x = a/2}$

$f'' = -2 < 0$ $\underline{\text{f concave}}$

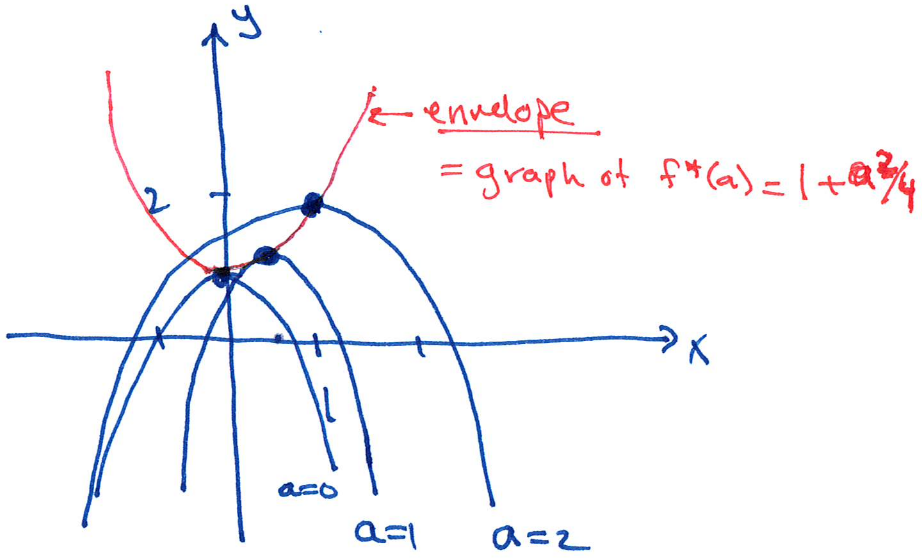
$f^*(a) = f(a/2)$
$= 1 + a \cdot (a/2) - (a/2)^2$
$= 1 + a^2/2 - a^2/4$
$= \underline{1 + a^2/4}$

$x^*(a) = \underline{a/2}$

Env. thm:

$$\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$$

$$\frac{2a}{4} = \underline{\frac{a}{2}}, \qquad x^*(a) = a/2$$



$\underline{a=1}$: $x^*(1) = 1/2 = 0.5$
$f^*(1) = 5/4 = 1.25$

$\underline{a=0}$: $x^*(0) = 0$
$f^*(0) = 1$

Summary: Envelope thms

Unconstrained case: $\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$

Lagrange case:
Kuhn-Tucker case: $\Big\}$ $\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$

$B = \underline{1}^T$ $\underline{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

(2) Minimum variance portfolios

$Bx = 1$ with $B = (1\ 1\ \dots\ 1)$

Ex: min $f(\underline{x}) = \underline{x}^T A \underline{x}$ when $x_1 + x_2 + \dots + x_n = 1$

when A is a symmetric $n \times n$-matrix that is positive definite.

$$L = \underline{x}^T A \underline{x} - \lambda(B\underline{x} - 1)$$

FOC: $L'(\underline{x}) = \begin{pmatrix} L'_{x_1} \\ L'_{x_2} \\ \vdots \\ L'_{x_n} \end{pmatrix} = 2A\underline{x} - \lambda(B^T) = \underline{0}$

$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$
$f'(\underline{x}) = 2A\underline{x} + B^T$
$H(f) = 2A$

C: $B\underline{x} = 1$

FOC: $2A\underline{x} = \lambda \cdot B^T \quad | : 2$

$A^{-1} \cdot | \quad A\underline{x} = \frac{\lambda}{2} \cdot B^T = \frac{\lambda}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda/2 \\ \lambda/2 \\ \vdots \\ \lambda/2 \end{pmatrix}$

$A^{-1}A\underline{x} = A^{-1} \frac{\lambda}{2} B^T$

$\underline{x} = \frac{\lambda}{2} A^{-1} B^T$

C: $B\underline{x} = B(\frac{\lambda}{2} A^{-1} B^T) = 1$

$\frac{\lambda}{2} \cdot B A^{-1} B^T = 1 \quad | \cdot 2$

Is A invertible?

A pos. defn.
$\Updownarrow$
$D_1, D_2, \dots, D_n > 0$
$\Updownarrow$
$\lambda_1, \lambda_2, \dots, \lambda_n > 0$
$\Downarrow$
$|A| = D_n > 0$
or
$|A| = \lambda_1 \lambda_2 \dots \lambda_n > 0$
Yes

$\lambda \cdot BA^{-1}B^T = 2$

$\frac{1}{BA^{-1}B^T} \cdot \Big|$ $\quad \lambda = \frac{2}{BA^{-1}B^T} > 0$

$\underline{x} = \frac{1}{BA^{-1}B^T} \cdot A^{-1}B^T$

$\underline{x}^T A^{-1} \underline{x}$ pos. defn. quadr. form

$\Downarrow$

$\underline{x} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ gives

$(1\ 1 \ldots 1)\, A^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} > 0$

$\parallel$

$BA^{-1}B^T > 0$

$BA^{-1}B^T = (1\ 1 \ldots 1) \cdot \begin{pmatrix} - & - \\ - & - \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$=$ a number

Note:

$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & \ldots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \ldots & C_{nn} \end{pmatrix}^T$

$n \times n$ matrix,

A^{-1} symmetric since

$C_{ij} = C_{ji}$

Ex: $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix}$

$C_{12} = \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix}$ $\quad C_{12} = \begin{vmatrix} 2 & 1 \\ 4 & 7 \end{vmatrix}$

A^{-1} is positive defn

Eigenval. of A:

$\lambda_1, \lambda_2, \ldots, \lambda_n > 0$

Eigenval. of A^{-1}:

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n} > 0$

$\Rightarrow A^{-1}$ pos. defn.

λ eigenvalue of A:

$A\underline{v} = \lambda \underline{v}$ for $\underline{v} \neq \underline{0}$

$A^{-1}A\underline{v} = A^{-1}\lambda\underline{v}$

$\underline{v} = \lambda \cdot A^{-1}\underline{v} \quad | : \lambda$

$\frac{1}{\lambda}\underline{v} = A^{-1}\underline{v}$

$\Rightarrow \frac{1}{\lambda}$ eigenvalue of A^{-1}

Conclusion:

A pos. defn $\Rightarrow$ One candidate pt

$(\underline{x}; \lambda) = \left(\frac{1}{BA^{-1}B^T} \cdot A^{-1}B^T, \frac{2}{BA^{-1}B^T} \right)$

SOC: $h(x) = \mathcal{L}\left(\underline{x}; \frac{2}{BA^{-1}B^T}\right)$

$= \underline{x}^T A \underline{x} - \frac{2}{BA^{-1}B^T} \cdot (B\underline{x} - 1)$

$H(h) = 2A$ pos. defn. since A is pos. defn.

$\Downarrow$

h convex

$\Downarrow$ SOC

$f_{min} = f\left(\frac{1}{BA^{-1}B^T} \cdot A^{-1}B^T\right)$ at $\underline{x} = \frac{1}{BA^{-1}B^T} \cdot A^{-1}B^T$

$$f_{min} = \underline{x}^T A \underline{x} \quad \text{when} \quad \underline{x} = \frac{1}{BA^{-1}B^T} \cdot A^{-1}B^T$$

$$= \underline{\frac{1}{BA^{-1}B^T}} \cdot (A^{-1}B^T)^T A \cdot \underline{\frac{1}{BA^{-1}B^T}} A^{-1}B^T$$

$$= \left(\frac{1}{BA^{-1}B^T}\right)^2 B \cdot (A^{-1})^T \cancel{A} \cdot \cancel{A^{-1}} B^T$$

$$= \left(\frac{1}{BA^{-1}B^T}\right)^{\cancel{2}} \cdot \cancel{BA^{-1}B^T} = \frac{1}{BA^{-1}B^T} > 0$$

See Section 6.7 in [E] for applications of these types of optimization problems to min. variance problems in portfolio th. / finance.