Plan

1. Kuhn-Tucker problems
2. Minimum variance portfolios ← No time today (will do examples next time)

① Kuhn-Tucker problems

Constrained optimization problems, where all constraints are closed inequalities ($\leq$, $\geq$) = Kuhn-Tucker problems.

Ex: max/min $f(x,y,z) = x+y+z$ (obj. function) when $4x^2+9y^2+z^2 \leq 36$ (constraints)

EVT: If f is cont. on a compact set D, then f has a max and a min on D

D: $4x^2+9y^2+z^2 \leq 36$ set of adm. pts.

Ex: $4x^2+9y^2+z^2 \leq 36$

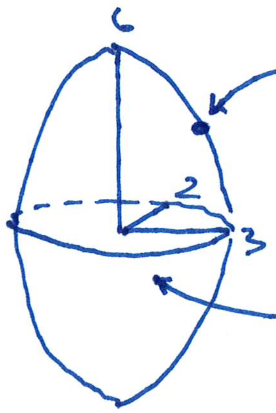
closed: ok

bounded: ok

$4x^2 \leq 36 \quad -3 \leq x \leq 3$

$9y^2 \leq 36 \quad -2 \leq y \leq 2$

$z^2 \leq 36 \quad -6 \leq z \leq 6$



Boundary pt: $4x^2+9y^2+z^2 = 36$

C binding

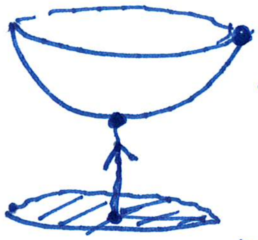
Interior pt: $4x^2+9y^2+z^2 < 36$

C non-binding

Method for finding candidates for max/min

Ex: a) Interior cand. pts $4x^2+9y^2+z^2<36$

Stationary pt. for f: $f'_x = f'_y = f'_z = 0$



$f = x^2+y^2$

$D: x^2+y^2 \leq 4$

$f'_x = 1 \neq 0$

$f'_y = 1 \neq 0$

$f'_z = 1 \neq 0$

$\lambda = 0$

no interior cand pts

b) Cand. pts that are boundary pts $4x^2+9y^2+z^2=36$

Method of Lagrange multipliers

$L = x+y+z-\lambda(4x^2+9y^2+z^2-36)$

FOC
$L'_x = 1-\lambda\cdot 8x = 0$
$L'_y = 1-\lambda\cdot 18y = 0$
$L'_z = 1-\lambda\cdot 2z = 0$

C: $4x^2+9y^2+z^2=36$

$\Rightarrow$ ... (see comp. in lecture 7) $\Rightarrow$

$(x,y,z;\lambda) = (9/7, 4/7, 36/7; 7/72)$, $f=7$

$(-9/7, -4/7, -36/7; -7/72)$, $f=-7$

$\lambda > 0$ max

$\lambda < 0$ min

Kuhn-Tucker formulation

Defn A Kuhn-Tucker problem is in standard form if it can be written as

$$\max f(\underline{x}) = f(x_1, x_2, \ldots, x_n) \text{ when } \begin{cases} g_1(\underline{x}) \leq a_1 \\ \vdots \\ g_m(\underline{x}) \leq a_m \end{cases}$$

Method for finding candidate pts

$$L(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) - \lambda_1 \cdot (g_1(\underline{x}) - a_1) - \cdots - \lambda_m (g_m(\underline{x}) - a_m)$$

FOC: $L'_{x_1} = 0$, $L'_{x_2} = 0$, $\ldots$, $L'_{x_n} = 0$

C: $g_1(\underline{x}) \leq a_1$, $\ldots$, $g_m(\underline{x}) \leq a_m$

CSC: $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_m \geq 0$ and $g_1(\underline{x}) < a_1 \Rightarrow \lambda_1 = 0$, $g_2(\underline{x}) < a_2 \Rightarrow \lambda_2 = 0$, $\ldots$, $g_m(\underline{x}) < a_m \Rightarrow \lambda_m = 0$

CSC = complementary slackness conditions

$g_i(\underline{x}) < a_i \Rightarrow \lambda_i = 0$ can also be written

$$\boxed{\lambda_i \cdot (g_i(\underline{x}) - a_i) = 0}$$

Idea: Solutions of the KT-conditions (FOC + C + CSC) = candidates for max.

SOC: If $(\underline{x}^*; \lambda^*)$ satisfy FOC+C+CSC, consider

$$h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$$

If h is concave, then $\underline{x}^*$ is max

Thm
If $\underline{x}^*$ is max in a KT problem, and NDCQ is satisfied at $\underline{x}^*$, then there is λ^* s.t. $(\underline{x}^*; \lambda^*)$ satisfies FOC+C+CSC.

Ex: max $f = x+y+z$ when $4x^2+9y^2+z^2 \le 36$ (std. form)

$L = x+y+z - \lambda(4x^2+9y^2+z^2-36)$

FOC: $L'_x = 1-\lambda\cdot 8x = 0$
$L'_y = 1-\lambda\cdot 18y = 0$
$L'_z = 1-\lambda\cdot 2z = 0$

C: $4x^2+9y^2+z^2 \le 36$

CSC: $\lambda \ge 0$ and $\lambda(4x^2+9y^2+z^2-36) = 0$

B: $4x^2+9y^2+z^2 = 36$	NB: $4x^2+9y^2+z^2 < 36$
$\lambda \ge 0$	$\lambda = 0$
$1-\lambda\cdot 8x=0$, $1-\lambda\cdot 18y=0$, $1-\lambda\cdot 2z=0$	$1=0$, $1=0$, $1=0$
$x=\frac{1}{8\lambda}$, $y=\frac{1}{18\lambda}$, $z=\frac{1}{2\lambda}$	no cand. pts.
C ⟹ … → $(4/7, 9/7, 36/7;\ 7/72)$, ~~$(-4/7, -9/7, -36/7; -7/72)$~~	

Concl: One cand. pt: $(x,y,z;\lambda) = (4/7, 9/7, 36/7;\ 7/72)$

SOC: $h(x,y,z) = L(x,y,z, 7/72) = x+y+z - \frac{7}{72}\cdot(4x^2+9y^2+z^2-36)$

$H(h) = -\frac{7}{72}\cdot 2\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ neg. def. $\Rightarrow h$ concave $\overset{SOC}{\Rightarrow}$ $f_{max} = 7$

NDCQ in the KT case

$$\left.\begin{matrix} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{matrix}\right\} \quad J = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 & \cdots & \partial g_1/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial g_m/\partial x_1 & \partial g_m/\partial x_2 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

$\underline{x}$ adm. $\quad J(\underline{x}) =$ keep the rows for B constr.
delete — " — NB — " —

NDCQ: rk $J(\underline{x})$ is maximal

Ex: $4x^2+9y^2+z^2 \le 36 \qquad J = (8x \;\; 18y \;\; 2z)$

$\underline{x}$:

- B: $(4x^2+9y^2+z^2=36)$ — $\boxed{\text{rk}\,(8x \;\; 18y \;\; 2z) = 1}$
- NB $(4x^2+9y^2+z^2<36)$ — No condition

Adm pts where NDCQ fails:

None!

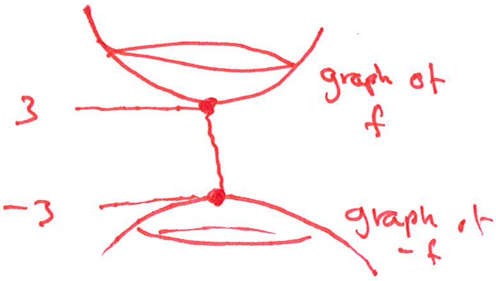
$8x=0 \quad 18y=0 \quad 2z=0$
$x=0 \quad y=0 \quad z=0$
$4x^2+9y^2+z^2=36$
$0 \ne 36$

What if the KT problem is not in std form.

Ex: $\min f = x^2+y^2+z^2$ when $\begin{cases} x+y+z \le 4 \\ x-y+3z \ge 9 \end{cases}$

change it into std form: $x-y+3z \ge 9 \quad | \cdot (-1)$

$$-x+y-3z \le -9$$



$$\min f = x^2+y^2+z^2$$
$$= \max -f = -x^2-y^2-z^2$$

Ex: $\max f = x+y+z+w$ when $3x^2+2xy+8xz-2xw+y^2+4yz+2yw+7z^2+4w^2 \le 18$

$x+y+z+w = B\underline{x}$, $B = (1 \;\; 1 \;\; 1 \;\; 1)$

$g(x,y,z,w) = \underline{x}^T A \underline{x}$, $A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$, $18 = a$

$$L = B\underline{x} - \lambda(\underline{x}^T A \underline{x} - 18)$$

FOC: $\begin{pmatrix} L'_x \\ L'_y \\ L'_z \\ L'_w \end{pmatrix} = L'(\underline{x}) = B^T - \lambda \cdot 2A\underline{x} = \underline{0}$

C: $g(\underline{x}) \le 18$ **CSC:** $\lambda \ge 0$ and $\lambda \cdot (g(\underline{x}) - 18) = 0$

(a) **Non-binding case:** $g(\underline{x}) < 18$, $\lambda = 0$, $B^T = \underline{0}$ impossible $\Rightarrow$ no crit. pts.

(b) Binding case:

$$\boxed{\begin{array}{l} g(\underline{x}) = 18 \ \checkmark \\ \lambda \geq 0 \\ B^T - 2\lambda A\underline{x} = \underline{0} \ \checkmark \end{array}}$$

$B^T = 2\lambda A\underline{x}$, $\lambda \neq 0$ $(\lambda > 0)$

$\frac{1}{2\lambda} B^T = A\underline{x}$

$$A\underline{x} = \frac{1}{2\lambda}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\lambda\\1/2\lambda\\1/2\lambda\\1/2\lambda\end{pmatrix}$$

$t = \frac{1}{2\lambda}$

use Gauss

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & -1 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ -1 & 1 & 0 & 4 & t \end{array}\right) \to \left(\begin{array}{cccc|c} \boxed{-1} & 1 & 0 & 4 & t \\ 1 & 1 & 2 & 1 & t \\ 4 & 2 & 7 & 0 & t \\ 3 & 1 & 4 & -1 & t \end{array}\right) \begin{array}{l}]1 \\ ]{+4} \\ ]{+3} \end{array}$$

$$\to \left(\begin{array}{cccc|c} \boxed{-1} & 1 & 0 & 4 & t \\ 0 & \boxed{2} & 2 & 5 & 2t \\ 0 & 6 & 7 & 16 & 5t \\ 0 & 4 & 4 & 11 & 4t \end{array}\right) \begin{array}{l}]{-3} \\ ]{-2} \end{array} \to \left(\begin{array}{cccc|c} x & y & z & w & \\ \boxed{-1} & 1 & 0 & 4 & t \\ 0 & \boxed{2} & 2 & 5 & 2t \\ 0 & 0 & \boxed{1} & 1 & -t \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

$\underline{w = 0}$ $\quad$ $\underline{z = -t}$ $\quad$ $2y + 2z + 5\cdot 0 = 2t$

$2y = 2t + 2t = 4t$ $\quad$ $\underline{y = 2t}$

$-x + 2t + 4\cdot 0 = t$ $\quad$ $-x = -t$ $\quad$ $\underline{x = t}$

$$\underline{x} = \begin{pmatrix}t\\2t\\-t\\0\end{pmatrix} = t\cdot\begin{pmatrix}1\\2\\-1\\0\end{pmatrix} \qquad g(\underline{x}) = \underline{x}^T A \underline{x} = t\,(1\ 2\ {-1}\ 0)\,A\,t\cdot\begin{pmatrix}1\\2\\-1\\0\end{pmatrix}$$

$= t^2\, g(1,2,-1,0) = 18 \Rightarrow$ $\quad$ $2t^2 = 18$

$g(1,2,-1,0) = 3\cdot 1^2 + 2\cdot 1\cdot 2 + 8\cdot 1\cdot(-1) + 2^2 + 4\cdot 2\cdot(-1) + 7\cdot 1$ $\quad$ $t^2 = 9$

$= 3 + 4 - 8 + 4 - 8 + 7 = 2$ $\quad$ $t = \pm 3$

$t = \pm 3 \quad t = \frac{1}{2\lambda} \Rightarrow 2\lambda = \frac{1}{\pm 3} = \frac{1}{3} \quad \lambda = \frac{1}{6}$

$(\lambda \geq 0)$

Candidate pts: $(x, y, z; w; \lambda) = (3, 6, -3, 0; 1/6)$

$f = 6$

SOC: $h(\underline{x}) = L(\underline{x}; 1/6) = x + y + z + w - \frac{1}{6}(\underline{x}^T A \underline{x} - 18)$

$H(h) = -\frac{1}{6} \cdot 2A = -\frac{1}{3} A$ neg. defn. $\Rightarrow h$ concave

$\Rightarrow \underline{x}^* = (3, 6, -3, 0)$ is max pt

$f_{max} = 6$

$$A = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 1 & 1 & 2 & 1 \\ 4 & 2 & 7 & 0 \\ -1 & 1 & 0 & 4 \end{pmatrix}$$

$D_1 = 3$

$D_2 = 2$

$D_3 = 4(-2) - 2(2) + 7 \cdot 2 = 2$

$D_4 = |A| = 2$ (see prev. page)

A pos. defn.

Alt: ("without" matrices) $L = x + y + z + w - \lambda(3x^2 + 2xy + 8xz - 2xw + y^2 + 4yz + 2yw + 7z^2 + 4w^2 - 18)$

FOC:

$L'_x = 1 - \lambda(6x + 2y + 8z - 2w) = 0$

$L'_y = 1 - \lambda(2x + 2y + 4z + 2w) = 0$

$L'_z = 1 - \lambda(8x + 4y + 14z) = 0$

$L'_w = 1 - \lambda(-2x + 2y + 8w) = 0$

C: $3x^2 + \ldots + 4w^2 \leq 18$

CSC: $\lambda \geq 0$

$\lambda \cdot (3x^2 + \ldots + 4w^2 - 18) = 0$

NB: $g(\underline{x}) < 18$, $\lambda = 0$ impossible

B: $g(\underline{x}) = 18$, $\lambda \geq 0$ FOC: $6x + 2y + 8z - 2w = 1/\lambda$

$2x + 2y + 4z + 2w = 1/\lambda$

$\vdots$

should use matrices and Gauss!

What about EVT instead of SOC?

$g(x,y,z,w) = 3x^2 + \_\_ + 4w^2 \leq 18$

$\underline{x}^T A \underline{x}$ = $\lambda_1 u_1^2 + \lambda_2 u_2^2 + \lambda_3 u_3^2 + \lambda_4 u_4^2 \leq 18$

A symm. is bounded

$\Rightarrow$ orth. diag. of A

$\lambda_1 u_1^2 \leq 18$

A pos. defn.

$u_1^2 \leq 18/\lambda_1$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$

$-\sqrt{18/\lambda_1} \leq u_1 \leq \sqrt{18/\lambda_1}$

$\vdots$

D is compact (closed + bounded)

$\underset{\text{EVT}}{\Rightarrow}$ there is a max

Ord. cond. pt: $(x,y,z,w;\lambda) = (3,6,-3,0;1/6)$ $f=6$

Adm pts where NDCQ fails:

Binding case: $g(\underline{x}) = 18$

no pts. $rk(g'_x \; g'_y \; g'_z \; g'_w) < 1$

$\Rightarrow g'_x = g'_y = g'_z = g'_w = 0$

$\Rightarrow x = y = z = w = 0$ (since $|A| \neq 0$)

but $g(0,0,0,0) = 0 \neq 18$

Nonbinding case:

no pts no cond. to check.

$\Rightarrow f_{max} = 6$ at $(3,6,-3,0)$ with $\lambda = 1/6$ by EVT