① Plenary Session 2: Mon
② Mock midterm exam
③ Midterm Evaluation

Plan

1 Constrained optimization: Lagrange problems
2 Second order conditions

Review: Unconstrained optimization
max/min $f(\underline{x}) = f(x_1, \dots, x_n)$

a) stationary pts:
$f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0$

b) classification:
$\underline{x}^*$: stationary pt

$$\boxed{\begin{array}{lll} H(f)(\underline{x}^*) \text{ pos. defn.} & \Rightarrow \underline{x}^* & \text{local min} \\ \quad \text{"} \quad \text{neg. defn} & \Rightarrow \text{"} & \text{local max} \\ \quad \text{"} \quad \text{indefn.} & \Rightarrow \text{"} & \text{saddle pt} \end{array}}$$

(second derivative test)

c) global max/min

$\boxed{f \text{ convex} \Updownarrow H(f)(\underline{x}) \text{ pos. semidefn. for all } \underline{x}}$ $\Rightarrow$ all stat. pts are global min

$\boxed{f \text{ concave} \Updownarrow H(f)(\underline{x}) \text{ neg. semidefn. for all } \underline{x}}$ $\Rightarrow$ all stat. pts. are global max

Otherwise:
- ad hoc (try to find counterex.)
- composite fn's where the kernel is convex/concave

Useful formulas:
When f is polynomial of deg. ≤ 2, hence
$f(\underline{x}) = \underline{x}^T A \underline{x} + B \underline{x} + C$
Then:
i) $f'(\underline{x}) = \begin{pmatrix} f'_{x_1} \\ \vdots \\ f'_{x_n} \end{pmatrix} = 2A\underline{x} + B^T$
ii) $H(f) = (f''_{x_i x_j}) = 2A$

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① Constrained optimization problems: Lagrange problems

max/min $f(\underline{x}) = f(x_1, \ldots, x_n)$ when certain constraints on $\underline{x} = x_1, \ldots, x_n$ are satisfied

Lagrange problem: special case of equality contraints.

Ex: max/min $\underbrace{f(x,y,z) = x+y+z}_{\text{objective fn.}}$ when $\underbrace{4x^2+9y^2+z^2=36}_{\text{equality constraint}}$

Defn: admissible points = points that satisfy all constraints.

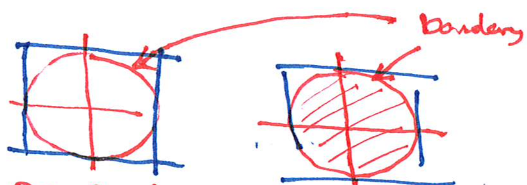
D = set of all adm. pts, subset of $\mathbb{R}^n$

D is closed in all Lagrange problems

In general:

Defn: D is closed if it contains all its boundary points

Note: $\boxed{\text{If all constraints are given by } =, \leq, \geq \text{ then D is closed}}$



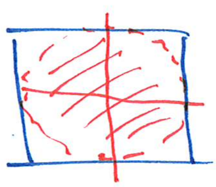
D: $x^2+y^2=1$ closed and bounded

D: $x^2+y^2 \leq 1$ closed and bounded

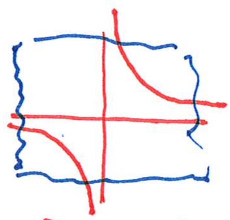
Defn. D is bounded if there are numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that

$$a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \ldots, a_n \leq x_n \leq b_n$$

for all $\underline{x} = (x_1, \ldots, x_n)$ in D.



D: $x^2+y^2<1$ not closed, bounded



D: $xy=1$ closed, not bounded

Defn. D is compact if it closed and bounded

Thm (Extreme Value thm = EVT)
If f is cont. on a compact set D, then f has a max and min on D.

Ex: $D: 4x^2+9y^2+z^2=36$

Bounded: $a_1 \le x \le b_1$
$a_2 \le y \le b_2$
$a_3 \le z \le b_3$

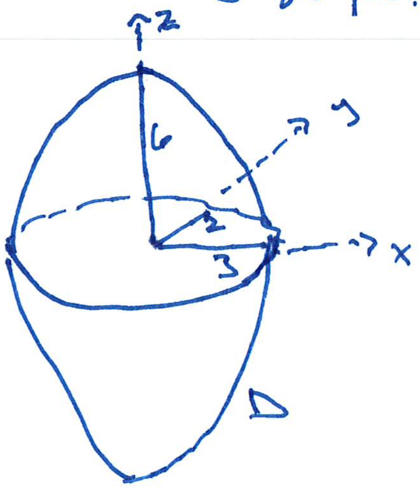
$4x^2 \le 36 \quad x^2 \le 9$
$9y^2 \le 36 \quad y^2 \le 4$
$z^2 \le 36$

$$\boxed{\begin{array}{c} -3 \le x \le 3 \\ -2 \le y \le 2 \\ -6 \le z \le 6 \end{array}}$$

D is bounded

max/min $f = x+y+z$ when $4x^2+9y^2+z^2=36$

there is a max/min in this Lagrange pb. $\Longleftarrow$ $D: 4x^2+9y^2+z^2=36$ is bounded



$f = x+y+z \to$ graph: $w = f(x,y,z)$ in $\mathbb{R}^4$

$f = 100$
$x+y+z=100$ level surface = plane
$\Downarrow$
Find value of c such that the plane $x+y+z=c$ just touches the surface D.

Method of Lagrange multipliers

for finding candidates for max/min

<u>In general:</u> max/min $f(x_1,\dots,x_n)$ when $\begin{cases} g_1(x_1,\dots,x_n)=a_1 \\ g_2(x_1,\dots,x_n)=a_2 \\ \vdots \\ g_m(x_1,\dots,x_n)=a_m \end{cases}$

<u>Lagrangian:</u>

$$\mathcal{L}(x_1,\dots,x_n;\lambda_1,\lambda_2,\dots,\lambda_m)=f(\underline{x})-\lambda_1(g_1(\underline{x})-a_1)-\lambda_2(g_2(\underline{x})-a_2)$$
$$\dots-\lambda_m(g_m(\underline{x})-a_m)$$

$\lambda_1,\lambda_2,\dots,\lambda_m$: Lagrange multipliers

<u>FOC:</u> $\mathcal{L}'_{x_1}=0$, $\mathcal{L}'_{x_2}=0$, $\vdots$, $\mathcal{L}'_{x_n}=0$ <u>C:</u> $g_1(\underline{x})=a_1$, $g_2(\underline{x})=a_2$, $\vdots$, $g_m(\underline{x})=a_m$

Lagrange conditions (FOC + C)

n+m equations in n+m variables

<u>Idea:</u> Candidate pts for max/min = all $(x_1,x_2,\dots,x_n;\lambda_1,\dots,\lambda_m)$ that satisfy FOC+C.

<u>Ex:</u> max/min $f=x+y+z$ when $\underbrace{4x^2+9y^2+z^2}_{g(x,y,z)}=\underbrace{36}_{a}$

$$\mathcal{L}=x+y+z-\lambda(4x^2+9y^2+z^2-36)$$

<u>FOC:</u>
$$\mathcal{L}'_x=1-\lambda\cdot 8x=0 \qquad 8\lambda x=1 \Rightarrow x=\frac{1}{8\lambda}\quad(\lambda\neq 0)$$
$$\mathcal{L}'_y=1-\lambda\cdot 18y=0 \qquad 18\lambda y=1 \qquad y=\frac{1}{18\lambda}$$
$$\mathcal{L}'_z=1-\lambda\cdot 2z=0 \qquad 2\lambda z=1 \qquad z=\frac{1}{2\lambda}$$

<u>C:</u>
$$4x^2+9y^2+z^2=36$$

$$4\left(\frac{1}{8\lambda}\right)^2+9\cdot\left(\frac{1}{18\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=36 \quad \Big|\cdot(2\lambda)^2$$

$$4\cdot\frac{1\cdot\cancel{(2\lambda)^2}}{4^2\cancel{(2\lambda)^2}} + 9\cdot\frac{1\cdot\cancel{(2\lambda)^2}}{9^2\cancel{(2\lambda)^2}} + \frac{\cancel{(2\lambda)^2}}{\cancel{(2\lambda)^2}} = 36\cdot(2\lambda)^2$$

$$\frac{1}{4}+\frac{1}{9}+1 = 36\cdot(2\lambda)^2 \quad \big|\cdot 36$$

$$9+4+36 = 36^2(2\lambda)^2 \quad \big|:36^2$$

$$\frac{49}{36^2} = (2\lambda)^2 \to 2\lambda = \pm\sqrt{\frac{49}{36^2}} = \pm\frac{7}{36} \qquad \underline{\lambda = \pm\frac{7}{72}}$$

$\underline{\lambda = 7/72:}$

$$x = \frac{1}{\lambda}\cdot\frac{1}{8} = \frac{\cancel{72}^{\,9}}{7}\cdot\frac{1}{\cancel{8}} = 9/7$$

$$y = \frac{1}{\lambda}\cdot\frac{1}{18} = \frac{72}{7}\cdot\frac{1}{18} = 4/7$$

$$z = \frac{1}{\lambda}\cdot\frac{1}{2} = \frac{72}{7}\cdot\frac{1}{2} = 36/7$$

$\underline{\lambda = -7/72:}$ $x = -9/7$, $y = -4/7$, $z = -36/7$

<u>Two candidate pts:</u> $(x,y,z;\lambda) = \left(9/7, 4/7, 36/7; \frac{7}{72}\right)$, $f = \frac{49}{7} = \underline{7}$

$\left(-9/7, -4/7, -36/7; -\frac{7}{72}\right)$ $f = -\frac{49}{7} = \underline{-7}$

$\Downarrow$ EVT

$f_{max} = \underline{\underline{7}}$ at $(x,y,z) = \underline{\underline{(9/7, 4/7, 36/7)}}$ with $\lambda = 7/72$

$f_{min} = \underline{\underline{-7}}$ at " $\underline{\underline{(-9/7, -4/7, -36/7)}}$ " $\lambda = -7/72$

<u>Thm,</u>

If $\underline{x}^* = (x_1^*, \ldots, x_n^*)$ is a max/min in a Lagrange problem, and $\underline{x}^*$ satisfies NDCQ, then there is $\underline{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*)$ such that $(\underline{x}^*; \underline{\lambda}^*)$ satisfies the Lagrange conditions FOC+C.

NDCQ = non-degenerate constraint qualification
(technical cond, and almost always satisfied)

Constraints:

$$\left.\begin{array}{l} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{array}\right\} \rightsquigarrow J = \begin{pmatrix} \partial g_1/\partial x_1 & \cdots & \partial g_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial g_m/\partial x_1 & \cdots & \partial g_m/\partial x_n \end{pmatrix}$$

Jacobian matrix

NDCQ: $\operatorname{rk} J = m$

Ex:

$4x^2+9y^2+z^2=36$ $\quad J = \begin{pmatrix} 8x & 18y & 2z \end{pmatrix}$

1×3 - matrix

NDCQ: $\operatorname{rk} J = 1$

NDCQ fails: $\operatorname{rk} J = 0$

$8x=0 \quad 18y=0 \quad 2z=0$

$x=0 \quad y=0 \quad z=0$

not admissible

Exceptional candidate pts:

$\operatorname{rk} J < m$ and C

In Ex: $\boxed{\text{No exceptional candidate pts}}$

Summary: In a Lagrange-problem, the candidate pts for max/min are:

(1) All $(x_1,\dots,x_n;\lambda_1,\dots,\lambda_m)$ that satisfy the Lagrange conditions FOC+C = ordinary candidate points

(2) Points $(x_1,\dots,x_n)$ that does not satisfy NDCQ but are admissible (satisfy C) = exceptional candidate points

② Second order condition

Thm (SOC)

If $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC+C in a Lagrange problem, then consider $h(x_1, \dots, x_n) = \mathcal{L}(x_1, \dots, x_n; \lambda_1^*, \dots, \lambda_m^*)$

$$= f(x_1, \dots, x_n) - \lambda_1^* \cdot (g_1(x_1, \dots, x_n) - a_1) - \dots$$

We have:

h concave $\Rightarrow$ $(\underline{x}^*; \underline{\lambda}^*)$ is max in the Lagrange pb.

h convex $\Rightarrow$ — " — min — " —

Ex. 1: max/min $f = x + y + z$ when $4x^2 + 9y^2 + z^2 = 36$

Ord. candidate pts:

$(9/7, 4/7, 36/7; 7/72) \longrightarrow h = \cancel{x+y+z} - \frac{7}{72}(4x^2 + 9y^2 + z^2 \cancel{- 36})$

$(-9/7, -4/7, -36/7; -7/72)$ $\quad H(h) = \underbrace{-\frac{7}{72} \cdot 2 \cdot \underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{pos. def.}}}_{\text{neg. def.}}$

$(9/7, 4/7, 36/7)$ is max pt. $\Longleftarrow$ h concave

$f_{max} = f(9/7, 4/7, 36/7) = \underline{\underline{7}}$

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Ex: max/min $f = 4x^2 + 9y^2 + z^2$ when $x+y+z = 49$

D = a plane in $\mathbb{R}^3$
not bounded

$L = 4x^2 + 9y^2 + z^2 - \lambda(x+y+z-49)$

FOC:
$$\boxed{\begin{aligned} L'_x &= 8x - \lambda = 0 \\ L'_y &= 18y - \lambda = 0 \\ L'_z &= 2z - \lambda = 0 \\ x+y+z &= 49 \end{aligned}}$$
C:

$x = \lambda/8 = 9$

$y = \lambda/18 = 4$

$z = \lambda/2 = 36$

$\frac{\lambda}{8} + \frac{\lambda}{18} + \frac{\lambda}{2} = 49 \quad | \cdot 2^3 \cdot 3^2 = \cdot 72$

$9\lambda + 4\lambda + 36\lambda = 49 \cdot 72$

$\cancel{49}\lambda = \cancel{49} \cdot 72$

$\underline{\lambda = 72}$

Ord. cand. pts: $(x,y,z;\lambda) = \underline{(9,4,36;72)}$

SOC: $h = 4x^2 + 9y^2 + z^2 - 72(\cancel{x+y+z-49})$

$H(h) = 2 \cdot \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ pos defn.

$\underline{h \text{ convex}} \xRightarrow{\text{SOC}} (9,4,36)$ is min. pt.

$f_{min} = f(9,4,36)$

$= 4 \cdot 9^2 + 9 \cdot 4^2 + 36^2$

$= 36(9 + 4 + 36) = 36 \cdot 49$

$\underline{\underline{= 1764}}$

there is no max

NDCQ:

$x+y+z=49$
$J = (1 \ 1 \ 1)$
$\text{rk } J = 1$

no adm pts where NDCQ fails $\Rightarrow (9,4,36;72)$ unique cand. pt for max/min

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Expl: SOC for max, i.e h is concave

$h = L(x_1, \dots, x_n ; \lambda_1^*, \dots, \lambda_m^*)$

Remark 1: $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC+C $\Rightarrow$ $\underline{x}^*$ is a stationary pt of h since

$$h'_{x_1}(\underline{x}^*) = L'_{x_1}(\underline{x}^*; \lambda^*) = 0$$
$$\vdots$$
$$h'_{x_n}(\underline{x}^*) = L'_{x_n}(\underline{x}^*; \lambda^*) = 0$$

Remark 2: $\underline{x}^*$ global max for h since h is concave

Remark 3: $L(\underline{x}^*; \lambda^*) = h(\underline{x}^*) \geq h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$

$$L(\underline{x}^*; \lambda^*) = f(\underline{x}^*) - \lambda_1^*(g_1(\underline{x}) - a_1) + \dots = f(\underline{x}^*)$$

$$L(\underline{x}; \underline{\lambda}^*) = f(\underline{x}) - \lambda_1^*(g_1(\underline{x}) - a_1) - \dots = f(\underline{x}) \quad \underline{x} \text{ adm.}$$

$\Rightarrow$ For all $\underline{x}$ that are adm. $f(\underline{x}^*) \geq f(\underline{x})$

$\Rightarrow$ $\underline{x}^*$ global max in Lagrange pb.

Ex: max/min $f = xw - yz$ when $\begin{cases} x^2+4y^2=4 \\ 4z^2+9w^2=36 \end{cases}$

$$L = xw - yz - \lambda_1(x^2+4y^2-4) - \lambda_2(4z^2+9w^2-36)$$

(1) $L'_x = w - \lambda_1 \cdot 2x = 0$

(2) $L'_y = -z - \lambda_1 \cdot 8y = 0$

(3) $L'_z = -y - \lambda_2 \cdot 8z = 0$

(4) $L'_w = x - \lambda_2 \cdot 18w = 0$

(5) $x^2+4y^2 = 4$

(6) $4z^2+9w^2 = 36$

$w = 2\lambda_1 x$ (1)

$z = -8\lambda_1 y$ (2)

$-y - 8\lambda_2(-8\lambda_1 y) = 0$ (3)

$x - 18\lambda_2(2\lambda_1 x) = 0$

$x(1-36\lambda_1\lambda_2) = 0$

(1)+(4): $\boxed{x=0 \text{ or } \lambda_1\lambda_2 = 1/36}$

$-y(1-64\lambda_1\lambda_2) = 0$

Four cases (a) – (d) based on $\rightarrow$ (2)+(3): $\boxed{y=0 \text{ or } \lambda_1\lambda_2 = 1/64}$

(a) $x=0, y=0$: $w=0$ from (1), $z=0$ from (2) $(x,y,z,w) = (0,0,0,0)$ does not satisfy C (5) and (6)

(b) $x=0, \lambda_1\lambda_2 = 1/64$: $w=0$ from (1)

(5) $x^2+4y^2 = 4y^2 = 4 \Rightarrow y^2=1$ $y=\pm 1$

(6) $4z^2+9w^2 = 4z^2 = 36$ $z^2=9$ $z=\pm 3$

(2) $\lambda_1 = -\frac{z}{8y} = \pm\frac{3}{8}$

$\lambda_1\lambda_2 = \frac{1}{64}$: $\lambda_2 = \pm\frac{1}{24}$

$\frac{1}{64} : \frac{3}{8} = \frac{1}{64}\cdot\frac{8}{3} = \frac{1}{8\cdot 3} = \frac{1}{24}$

Candidate pts:

$(0,1,3,0;\ -3/8, -1/24)$ $f=-3$

$(0,-1,-3,0;\ -3/8, -1/24)$ $f=-3$

$(0,1,-3,0;\ 3/8, 1/24)$ $f=3$

$(0,-1,3,0;\ 3/8, 1/24)$ $f=3$

(c) $y=0,\ \lambda_1\lambda_2=\frac{1}{36}$: $z=0$ from (2)

(5) $x^2=4 \quad x=\pm 2$

(6) $9w^2=36 \quad w^2=4 \quad w=\pm 2$

(1) $\lambda_1=\frac{w}{2x}=\pm\frac{1}{2}$

$\lambda_1\lambda_2=\frac{1}{36}$: $\lambda_2=\pm\frac{1}{18}$ $\qquad \frac{1}{36}:\frac{1}{2}=\frac{1}{36}\cdot\frac{2}{1}=\frac{1}{18}$

Cand. pts:

$(2,0,0,2;\ 1/2,\ 1/18)$ $f=4$ $\qquad (2,0,0,-2;\ -1/2,\ -1/18)$ $f=-4$

$(-2,0,0,-2;\ 1/2,\ 1/18)$ $f=4$ $\qquad (-2,0,0,2;\ -1/2,\ -1/18)$ $f=-4$

(d) $\lambda_1\lambda_2=\frac{1}{36},\ \lambda_1\lambda_2=\frac{1}{64}$: impossible

SOC on $(2,0,0,2)$ $\lambda_1=\frac{1}{2}$ $\lambda_2=\frac{1}{18}$ $\quad h = xw - yz - \frac{1}{2}(x^2+4y^2-4) - \frac{1}{18}(4z^2+9w^2-36)$

$f_{max}=f(2,0,0,2)=4$
$=f(-2,0,0,-2)=4$

$$H(h) = 2\underbrace{\begin{pmatrix} -1/2 & 0 & 0 & 1/2 \\ 0 & -2 & -1/2 & 0 \\ 0 & -1/2 & -2/9 & 0 \\ 1/2 & 0 & 0 & -1/2 \end{pmatrix}}_{A}$$

Similar computation using SOC:

$f_{min}=f(-2,0,0,2)=-4$
$=f(2,0,0,-2)=-4$

A:

$D_1=-1/2<0$

$D_2=1>0$

$D_3=-\frac{1}{2}(4/9-1/4)$
$=-\frac{1}{2}\left(\frac{16-9}{36}\right)<0$

$D_4=0$

RRC:

rk $A=3$

A is neg. semidefin.

$\Downarrow$

$H(h)=2A$ neg. semidefin.

$\Downarrow$

h concave

$\Downarrow$

$(2,0,0,2)$ is max.