Plan

1. Eigenvalues and eigenvectors
2. Diagonalization
3. Computing powers of matrices

Plenary Session I: Mon 16-19
A1-040

Review: Lecture 3
- determinants and minors
- computing rank using minors

① Eigenvalues and eigenvectors

A $n\times n$ matrix $\quad T(\underline{v}) = A\cdot\underline{v}$ ($\underline{v}$ n-vector) $\quad T: \mathbb{R}^n \to \mathbb{R}^n$, $\underline{v} \mapsto A\cdot\underline{v}$

(*) $\boxed{A\cdot\underline{v} = \lambda\cdot\underline{v}}$ (λ: number)

Ex: Unemployment

$\underline{v} = \begin{pmatrix} e \\ u \end{pmatrix}$ empl. vector

10% unempl.

$\underline{v} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$

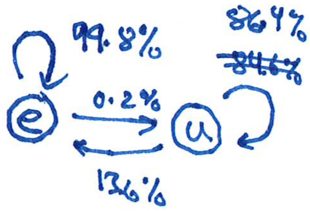
$e + u = 1$

$0 \le e, u \le 1$

$\underline{v}_{t+1} = \begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = A \cdot \begin{pmatrix} e_t \\ u_t \end{pmatrix}$

$$A = \begin{array}{cc} \begin{array}{cc} e & u \end{array} & \\ \begin{pmatrix} 0.998 & 0.136 \\ 0.002 & 0.864 \end{pmatrix} & \begin{array}{c} e \\ u \end{array} \end{array}$$

transition matrix



$$\begin{pmatrix} e_{t+1} \\ u_{t+1} \end{pmatrix} = \begin{pmatrix} 0.998\, e_t + 0.136\, u_t \\ 0.002\, e_t + 0.864\, u_t \end{pmatrix}$$

$\underline{v}_0 = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}$: $\quad \underline{v}_T = A^T \cdot \underline{v}_0 \quad (T \gg 0) \quad \rightsquigarrow \quad \lim_{T\to\infty} A^T \cdot \underline{v}_0$

Defn: An eigenvalue is a number λ such that (*) $A\cdot\underline{v} = \lambda\cdot\underline{v}$ has non-trivial soln's $\underline{v} \neq \underline{0}$. The eigenspace E_λ is the collection of all solutions $\underline{v}$ of (*), and any non-zero vector $\underline{v}$ in E_λ is called an eigenvector of A with eigenvalue λ.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$

$A\underline{v} = \begin{pmatrix} x + 2y \\ 2x + y \end{pmatrix} = \lambda\underline{v} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

$x - \lambda x + 2y = 0$

$2x + y - \lambda y = 0$

$$\boxed{\begin{aligned} (1-\lambda)x + 2y &= 0 \\ 2x + (1-\lambda)y &= 0 \end{aligned}}$$

homog. lin. sys. with parameter λ

Note: (*) $A\underline{v} = \lambda \underline{v}$

$$A\underline{v} - \lambda\underline{v} = \underline{0}$$

$$A\underline{v} - \lambda I \underline{v} = \underline{0}$$

(**) $(A - \lambda I)\underline{v} = \underline{0}$

homogeneous lin. sys. with parameter λ.

$$I = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & \ddots \end{pmatrix} \qquad \lambda I = \begin{pmatrix} \lambda & & 0 & \\ & \lambda & & \\ 0 & & \lambda & \\ & & & \ddots \end{pmatrix}$$

Result:

i) λ eigenvalue of A $\Leftrightarrow$ $|A - \lambda I| = 0$ (n'th order polyn. eqn) (char. eqn.)

ii) For each eigenvalue λ, $E_\lambda = \text{Null}(A - \lambda I)$ (lin. sys.)

Ex. 1:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(1-\lambda) - 2 \cdot 2 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\underline{\lambda_1 = -1}, \ \underline{\lambda_2 = 3}$$

E_{-1}: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad 2x + 2y = 0 \quad \underline{x = -y}$, $\underline{y \text{ free}}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{= \underline{v_1}}$ $\quad E_{-1} = \text{span}(\underline{v_1})$

E_3: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$ $\quad \underline{x = y}$, $\underline{y \text{ free}}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{= \underline{v_2}}$ $\quad E_3 = \text{span}(\underline{v_2})$

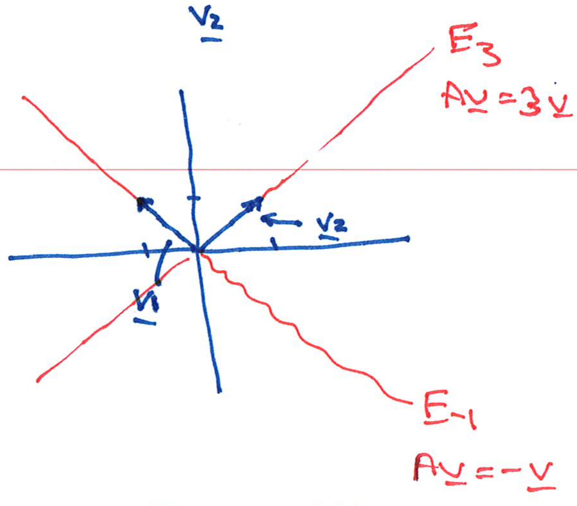
Note:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$\boxed{\lambda^2 - \text{tr}(A)\lambda + |A| = 0}$$



Ex 2:

$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\lambda^2 + 1 = 0$$

no eigenvalues

Ex. 3

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$\lambda_1 = \lambda_2 = 1$

$\underline{\lambda = 1 \ (\text{mult. } 2)}$

$\underline{E_1}$: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad y = 0$, x free

$\underline{E_1 = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Defn: If $|A-\lambda I|$ has $(\lambda-\lambda_i)^m$ as a factor, then $\lambda=\lambda_i$ has multiplicity m.

Ex: $|A-\lambda I| = (\lambda-2)^2\cdot(\lambda+3) \Rightarrow \begin{cases} \lambda=2 & \text{mult } m=2 \\ \lambda=-3 & \text{mult } m=1 \end{cases}$

Result:

i) If λ is an eigenvalue of mult. m, then $1 \leq \underset{\substack{\| \\ \#\text{ degrees of freedom}}}{\dim E_\lambda} \leq m$

ii) If $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the eigenvalues of A (counted with mult.), then

$$\boxed{\begin{aligned} \lambda_1+\lambda_2+\cdots+\lambda_n &= tr(A) \\ \lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_n &= |A| \end{aligned}}$$

iii) If A is symmetric, then A has n eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ (counted with mult.) and $\dim E_{\lambda_i}=m_i$ (where m_i is the multiplicity of λ_i).

Ex:

$A=\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

i) $\lambda=1$ is an eigenvalue: $A-\lambda I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\dim E_1 = 2 \leq m \Rightarrow \lambda_1=\lambda_2=1$

ii) $\underset{\substack{\| \\ 1}}{\lambda_1}+\underset{\substack{\| \\ 1}}{\lambda_2}+\lambda_3 = \underset{\substack{\| \\ 6}}{tr(A)}$ $\qquad \lambda_3=4$

Alternative:

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)\cdot\left((2-\lambda)^2-1^2\right) - 1\cdot\left((2-\lambda)-1\right) + 1\cdot\left(1-(2-\lambda)\right) = 0$

$(2-\lambda)\left(\lambda^2-4\lambda+3\right) - (1-\lambda) + (\lambda-1) = 0$

$(2-\lambda)\left((\lambda-3)(\lambda-1)\right) + (\lambda-1) + (\lambda-1) = 0$

$(\lambda-1)\cdot\left[(2-\lambda)(\lambda-3)+1+1\right] = 0$

$(\lambda-1)\cdot\left(-\lambda^2+5\lambda-6+2\right) = 0$

$$(\lambda-1)\cdot(-1)\cdot(\lambda^2-5\lambda+4)=0$$

$$-(\lambda-1)(\lambda-4)(\lambda-1)=0$$

$$-(\lambda-1)^2(\lambda-4)=0$$

$$\underline{\lambda_1=\lambda_2=1}\,,\ \underline{\lambda_3=4}$$

(2) Diagonalisation

A $n \times n$ matrix

Recall:

A diagonal matrix can be written

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & d_n \end{pmatrix}$$

Then

$$D^N = \begin{pmatrix} d_1^N & & 0 \\ & d_2^N & \\ 0 & & \ddots \; d_n^N \end{pmatrix}$$

Defn. The matrix A is diagonalizable if there is a diagonal matrix D and an invertible matrix P s.t. $P^{-1}AP = D$.

Remark: $P^{-1}AP = D \quad | P \cdot$

$AP = PD \quad | \cdot P^{-1} \rightarrow \boxed{A = PDP^{-1}}$

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$ $\quad P = \left(\underline{v_1} \,|\, \underline{v_2} \,|\, \cdots \,|\, \underline{v_n} \right)$

$$AP = A\left(\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n}\right) = \left(A\underline{v_1} | A\underline{v_2} | \cdots | A\underline{v_n}\right) = PD = \left(\underline{v_1} | \cdots | \underline{v_n}\right) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$= \left(\lambda_1 \underline{v_1} \,|\, \lambda_2 \underline{v_2} \,|\, \cdots \,|\, \lambda_n \underline{v_n}\right)$$

Equality holds: $AP = PD$

$\Updownarrow$

$A\underline{v_1} = \lambda_1 \underline{v_1}, \; A\underline{v_2} = \lambda_2 \underline{v_2}, \; \ldots, \; A\underline{v_n} = \lambda_n \cdot \underline{v_n}$

Result:

i) A is diagonalizable $\Leftrightarrow$ (1) A has n eigenvalues (counted w. mult.) and (2) A has n lin. independent eigenvector

ii) A has n distinct eigenvalues $\Rightarrow$ A diagonalizable

iii) A symmetric $\Rightarrow$ A diagonalizable

Ex:

$A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ $\lambda_1=-1$ $\underline{v_1}=\begin{pmatrix}-1\\1\end{pmatrix}$ $D=\begin{pmatrix}-1&0\\0&3\end{pmatrix}$ Diag: Yes

$\lambda_2=3$ $\underline{v_2}=\begin{pmatrix}1\\1\end{pmatrix}$ $P=\begin{pmatrix}-1&1\\1&1\end{pmatrix}$

$P^{-1}AP=D$

$AP=PD$

$A=PDP^{-1}$

$A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ no eigenval. $\lambda^2=-1$

$A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\lambda_1=1$, $\lambda_2=1$ $\longrightarrow$ $D=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ Diag: NO.

$\underline{\lambda=1}$ $(m=2)$ $\underline{v_1}=\begin{pmatrix}1\\0\end{pmatrix}$ $P=\begin{pmatrix}1&?\\0&?\end{pmatrix}$ Diag: No

$A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$ $\lambda=1$ (mult $=2$) $\lambda=4$ $D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}$ Diag: Yes

$P=\begin{pmatrix}-1&-1&1\\1&0&1\\0&1&1\end{pmatrix}$

E_1: $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\to\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$ $x=-y-z$, y,z free

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-y-z\\y\\z\end{pmatrix}=y\cdot\underbrace{\begin{pmatrix}-1\\1\\0\end{pmatrix}}_{=\underline{v_1}}+z\cdot\underbrace{\begin{pmatrix}-1\\0\\1\end{pmatrix}}_{=\underline{v_2}}$

$E_1=\text{span}(\underline{v_1},\underline{v_2})$

E_4: $\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}$ $\underline{v_3}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$E_4=\text{span}(\underline{v_3})$

Note: $\underset{n\times n}{A}$ diagonalizable $\Leftrightarrow$ A has n eigenvalues and (c. with mult.) $\dim E_\lambda=m$ for all eigenval. λ with mult. $m>1$.

③ Computing powers of matrices

$\underset{n\times n}{A}$: Want to compute A^N when $N \gg 0$ is big.

Assume that A has a diagonalization $P^{-1}AP = D$.

$AP = PD$

$A = PDP^{-1}$

$$A^N = (PDP^{-1})^N = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})}_{N \text{ times}}$$

$$= PD^NP^{-1}$$

$$= (\underline{v}_1 | \dots | \underline{v}_n) \cdot \begin{pmatrix} \lambda_1^N & & 0 \\ & \lambda_2^N & \\ 0 & & \ddots \; \lambda_n^N \end{pmatrix} \cdot (\underline{v}_1 | \dots | \underline{v}_n)^{-1}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad D^N = \begin{pmatrix} (-1)^N & 0 \\ 0 & 3^N \end{pmatrix}$

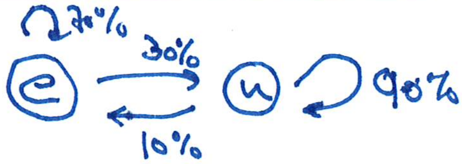
$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad P^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^N = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} (-1)^N & 0 \\ 0 & 3^N \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -(-1)^N & 3^N \\ (-1)^N & 3^N \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (-1)^N + 3^N & -(-1)^N + 3^N \\ -(-1)^N + 3^N & (-1)^N + 3^N \end{pmatrix}$$

$$= \underbrace{\frac{(-1)^N}{2}}_{\pm\frac{1}{2}} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \underbrace{\frac{3^N}{2}}_{\text{tends to } \infty \text{ as } N \text{ grows}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{pmatrix}$



Eigenvalues: $\lambda^2 - 1.6\lambda + 0.6 = 0$

$(\lambda-1)(\lambda-0.6) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 0.6$

$$\boxed{D = \begin{pmatrix} 1 & 0 \\ 0 & 0.6 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}}$$

Eigenspaces:

E_1: $\begin{pmatrix} -0.3 & 0.1 \\ 0.3 & -0.1 \end{pmatrix} \to \begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$E_1 = \text{span}(\underline{v_1})$

$E_{0.6}$: $\begin{pmatrix} 0.1 & 0.1 \\ 0.3 & 0.3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \qquad D^N = \begin{pmatrix} 1^N & 0 \\ 0 & 0.6^N \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.6^N \end{pmatrix}$

$$A^N = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0.6^N \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -0.6^N \\ 3 & 0.6^N \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 + 3\cdot 0.6^N & 1 - 0.6^N \\ 3 - 3\cdot 0.6^N & 3 + 0.6^N \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} + \frac{0.6^N}{4} \cdot \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix} \xrightarrow[N \to \infty]{} \begin{pmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{pmatrix}$$