**Plan**

1 Matrices and determinants
2 Minors and rank
3 Solving linear systems using minors

**Review Lecture 2**

- Solve vector eqn. using Gaussian elim.
- span and linear independence of vectors
- dimension and base of vector spaces, examples of vector spaces are column / null space of a matrix

$m \times n$ matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

$A$ is square if $m = n$

## ① Determinants

$A$ ($n \times n$ matrix) $\rightsquigarrow \det(A) = |A|$ determinant of $A$, a number

Methods for computing $\det(A)$

① Cofactor expansion
② Using Gaussian process

Ex:

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ : $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = +1 \cdot (18-12) - 1 (9-4) + 1 \cdot (3-2)$

$\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix}$ $\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix}$ $\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= 6 - 5 + 1 = \underline{\underline{2}}$

Sign: $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

minor: $M_{ij}$ = det. of the matrix you get when you delete row $i$, col $j$

$C_{ij} = (-1)^{i+j} \cdot M_{ij}$

**Fact:** Cofactor expansion along any row or column of $A$ gives $\det(A)$.

Ex:

$$+1\cdot\begin{vmatrix}1&3\\0&2\end{vmatrix}=+1\cdot 1\cdot 2$$

$$\begin{vmatrix}1&1&1\\1&2&4\\1&3&9\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&3\\0&2&8\end{vmatrix}=\begin{vmatrix}1&1&1\\0&1&3\\0&0&2\end{vmatrix}=1\cdot1\cdot2=\underline{\underline{2}}$$

Fact: $A \to B$ one elementary row operation.

i) $|B| = -|A|$ if we switch two rows
ii) $|B| = c\cdot|A|$ if we multiply a row by $c$
iii) $|B| = |A|$ if we add a multiple of one row to another row

Fact: The determinant of an upper triangular matrix is the product of the diag. entries

upper triangular: all entries under the main diagonal are zero

Notice: i) You can only compute determinants of square matrices
ii) Square echelon forms are always upper triangular

Ex:

$$\begin{pmatrix}1&2&1&0\\0&3&4&7\\0&0&3&1\\0&0&0&4\end{pmatrix}$$

echelon form
$\det = 1\cdot3\cdot3\cdot4 = \underline{36} \neq 0$

$$\underbrace{\begin{pmatrix}1&2&1&0\\0&3&4&7\\0&0&2&1\\0&0&0&0\end{pmatrix}\quad\begin{pmatrix}1&2&3&4\\0&0&7&1\\0&0&0&3\\0&0&0&0\end{pmatrix}}_{\text{echelon forms}}$$

$\det = 0$

$A$ $n \times n$ matrix

| $\|A\| \neq 0$ | $\|A\| = 0$ |
|---|---|
| * $\text{rk}\, A = n$ | * $\text{rk}\, A < n$ |
| * $A\underline{x} = \underline{b}$ has a unique solution for every $\underline{b}$ | * $A\underline{x} = \underline{b}$ has either no solutions or inf. many solutions for any $\underline{b}$. |
| * $A$ is invertible ($A^{-1} = \frac{1}{\|A\|} \cdot \text{adj}(A)$) | * $A$ is not invertible (there is no inverse) |
| * the rows of $A$ are linearly independent | * the rows of $A$ are linearly dependent |
| * the cols of $A$ are linearly independent | * the cols of $A$ are linearly dependent |

Recall: - any linear system with augmented matrix $(A|\underline{b})$ can be written $A\underline{x} = \underline{b}$, with $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

- An inverse $A^{-1}$ of $A$ is a matrix such that $\begin{cases} A \cdot A^{-1} = I \\ A^{-1} \cdot A = I \end{cases}$ where $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ identity matrix

and $A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}^T$

- useful formulas:

i) $|A \cdot B| = |A| \cdot |B|$ ii) $|A^T| = |A|$ iii) $(AB)^{-1} = B^{-1} \cdot A^{-1}$ iv) $(AB)^T = B^T \cdot A^T$

**Ex:**

$$A = \begin{vmatrix} 1 & 1 & 2 & 4 \\ 3 & 1 & 1 & 0 \\ 2 & 3 & 0 & 2 \\ 6 & 5 & 3 & 6 \end{vmatrix} \qquad |A| = \underline{0}$$

$-1$, $-1$

$R(4) = R(1) + R(2) + R(3)$

## ② Minors and rank

$A$ $m \times n$ matrix

**Defn:** An $r$-minor of $A$ is the determinant of an $r \times r$ submatrix of $A$

**Ex:** $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 4 \end{pmatrix}$

2-minors:

$$M_{12,12} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1-6 = -5$$

$$M_{12,23} = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 8-2 = 6$$

$$M_{12,13} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2$$

1-minors:

$1, 2, 2$
$3, 1, 4$

**Result:** The rank of a $m \times n$ matrix $A$ is maximal number $r$ such that $A$ has a non-zero $r$-minor.

$$\boxed{\text{rk}\,A < r \iff \text{all } r\text{-minors of } A \text{ are zero.}}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 3 & 0 & 2 \end{pmatrix}$

3-minors:

$M_{123,123} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 3 & 0 \end{vmatrix} = 0$

$M_{123,234} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{vmatrix} = -3$

$\text{rk}\, A = \underline{\underline{3}}$

$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$

$|A| = 1 \cdot (3) + 2(-3)$
$= \underline{-3} \neq 0$

$\text{rk}\, A = \underline{\underline{3}}$

Ex: $A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}} & \begin{matrix} 1 & 1 \\ -1 & 0 \end{matrix} \\ \begin{matrix} 3 & 3 \end{matrix} & \begin{matrix} 0 & 1 \end{matrix} \end{pmatrix}$ $R(3) = R(1) + R(2)$

all 3-minors $= 0 \Rightarrow \text{rk}\, A < 3$

2-minors: $M_{12,12} = 1 - 4 \neq 0 \Rightarrow \text{rk}\, A = \underline{\underline{2}}$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & p & q \end{pmatrix}$

2-minors: $p-1,\ q-p,\ q-1$

$p - 1 = 0 \quad p = 1$
$q - p = 0$
$q - 1 = 0 \quad q = 1$

$(p,q) = (1,1)$ : $\text{rk}\, A < 2$
$(p,q) \neq (1,1)$ : $\text{rk}\, A = 2$

Conclusion:

$$\text{rk}\, A = \begin{cases} 2 & , (p,q) \neq (1,1) \\ 1 & , (p,q) = (1,1) \end{cases}$$

## 3 Solving linear systems using minors

Ex:
$$\begin{aligned} x+y+z+w&=5\\ 2x-y+3z+4w&=2\\ 3x\phantom{+y}+4z+6w&=-1 \end{aligned}$$

$$A=\begin{pmatrix} 1 & 1 & 1 & 1\\ 2 & -1 & 3 & 4\\ 3 & 0 & 4 & 6 \end{pmatrix}$$

3-minors:

$$M_{123,123}=0$$

$$M_{123,124}=\begin{vmatrix} 1 & 1 & 1\\ 2 & -1 & 4\\ 3 & 0 & 6 \end{vmatrix}$$

$$=3\cdot(5)+6\cdot(-3)=-3\neq 0$$

$\Rightarrow \operatorname{rk} A = 3$

2-minors: $\underbrace{\begin{vmatrix} 1 & 1\\ 2 & -1 \end{vmatrix}}_{M_{12,12}} = -1-2=-3\neq 0$

$\Rightarrow$ one degree of freedom ($z$ is free), inf. many solutions

$$\Downarrow$$

$$\begin{aligned} x+y+w&=5-z\\ 2x-y+4w&=2-3z\\ 3x\phantom{-y}+6w&=-1-4z \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1\\ 2 & -1 & 4\\ 3 & 0 & 6 \end{pmatrix}\cdot\begin{pmatrix} x\\ y\\ w \end{pmatrix}=\begin{pmatrix} 5-z\\ 2-3z\\ -1-4z \end{pmatrix}$$

$$\begin{pmatrix} x\\ y\\ w \end{pmatrix}=\begin{pmatrix} 1 & 1 & 1\\ 2 & -1 & 4\\ 3 & 0 & 6 \end{pmatrix}^{-1}\cdot\begin{pmatrix} 5-z\\ 2-3z\\ -1-4z \end{pmatrix}$$

$$=\frac{1}{-3}\begin{pmatrix} -6 & 0 & 3\\ -6 & 3 & 3\\ 5 & -2 & -3 \end{pmatrix}^T\cdot\begin{pmatrix} 5-z\\ 2-3z\\ -1-4z \end{pmatrix}$$

$$\begin{pmatrix} x\\ y\\ w \end{pmatrix}=-\frac{1}{3}\begin{pmatrix} -6 & -6 & 5\\ 0 & 3 & -2\\ 3 & 3 & -3 \end{pmatrix}\begin{pmatrix} 5-z\\ 2-3z\\ -1-4z \end{pmatrix}=-\frac{1}{3}\begin{pmatrix} 4z-47\\ -z+8\\ 24 \end{pmatrix}=\begin{pmatrix} -4z/3+47/3\\ z/3-8/3\\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix}=\begin{pmatrix} -4z/3+47/3\\ z/3-8/3\\ z\\ -8 \end{pmatrix}$$

where $z$ is free

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