Plan

1 Span of vectors and linear independence
2 Vector spaces, bases and dimensions
3 Orthogonality and orthogonal projections

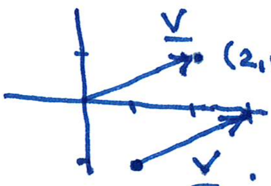
Lecture 1:

- Gaussian elimination
- geometry - rank
- homogeneous system
- nontrivial sol'ns

1 Span of vectors and linear independence

Ex: $\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

col. vector



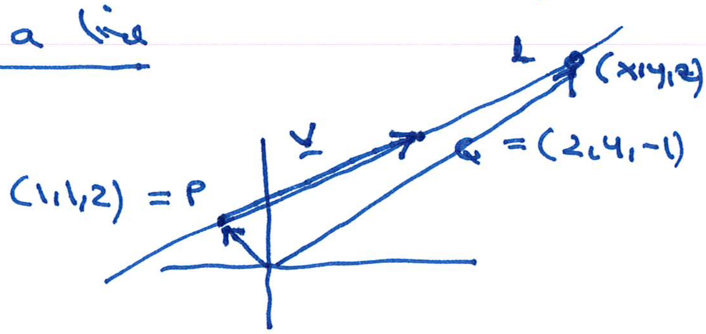
$\underline{v} = (2,1)$

$\|\underline{v}\| = \sqrt{2^2+1^2} = \sqrt{5}$

$\underline{v} = \vec{PQ} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

Parametric representation of a line

Ex: ℓ is the line through $(1,1,2)$ and $(2,4,-1)$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{OP} + t \cdot \underset{\substack{\| \\ \vec{PQ}}}{\underline{v}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1+t \\ 1+3t \\ 2-3t \end{pmatrix}}}$$

Span:

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ (n-vectors)

$c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + \ldots + c_r \cdot \underline{v_r}$ is called a <u>linear combination</u> of the vectors $\underline{v_1}, \ldots, \underline{v_r}$.

Defn:

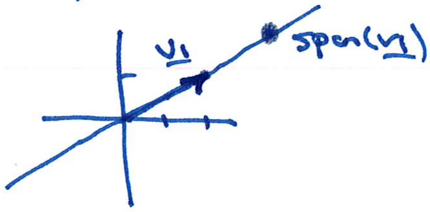
$\text{span}(\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r})$ = all lin. comb. $c_1 \underline{v_1} + \ldots + c_r \cdot \underline{v_r}$ for numbers $c_1, c_2, \ldots, c_r$

$= \{ c_1 \underline{v_1} + \ldots + c_r \underline{v_r} \mid c_1, \ldots, c_r \text{ are numbers in } \mathbb{R} \}$

Ex:

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

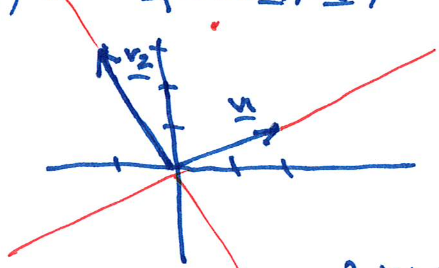
$\text{span}(\underline{v_1}) = \{ c_1 \underline{v_1} \mid c_1 \text{ is a number} \}$



Ex:

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$:

$\text{span}(\underline{v_1}, \underline{v_2}) = \{ c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} \mid c_1, c_2 \text{ numbers} \}$



Is $\text{span}(\underline{v_1}, \underline{v_2})$ = all 2-vectors?

$\begin{pmatrix} 7 \\ 7 \end{pmatrix} = 4 \underline{v_1} + 1 \cdot \underline{v_2}$

<u>Is $\begin{pmatrix} 7 \\ 7 \end{pmatrix}$ in $\text{span}(\underline{v_1}, \underline{v_2})$? Yes.</u>

$x_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

vector eqn.

$$\begin{aligned} 2x_1 - x_2 &= 7 \\ x_1 + 3x_2 &= 7 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= 7 \\ 7/2\, x_2 &= 7/2 \end{aligned} \qquad \begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}$$

$$\begin{array}{c} \left(\begin{array}{cc|c} 2 & -1 & 7 \\ 1 & 3 & 7 \end{array} \right) \\ \uparrow \quad \uparrow \\ \underline{v_1} \quad \underline{v_2} \end{array} \xrightarrow{-1/2} \left(\begin{array}{cc|c} 2 & -1 & 7 \\ 0 & 7/2 & 7/2 \end{array} \right)$$

Is $\begin{pmatrix} a \\ b \end{pmatrix}$ in span$(\underline{v_1}, \underline{v_2})$? $\quad \left(\begin{array}{cc|c} 2 & -1 & a \\ 1 & 3 & b \end{array}\right) \cdot (-\tfrac{1}{2}) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & * \\ 0 & 7/2 & * \end{array}\right)$

Yes. one solution

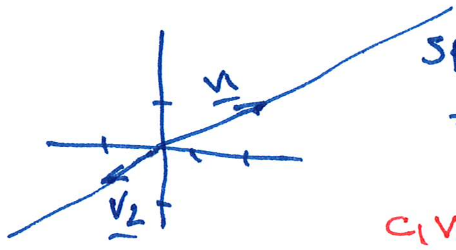
Conclusion: $\text{span}(\underline{v_1}, \underline{v_2}) = \mathbb{R}^2$ (all 2-vectors)

Ex:

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \underline{v_2} = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}$

$$\boxed{\underline{v_2} = -\tfrac{1}{2} \cdot \underline{v_1}}$$

$\underline{v_1} = -2\underline{v_2}$



$\text{span}(\underline{v_1}, \underline{v_2}) = \text{span}(\underline{v_1})$

$c_1\underline{v_1} + c_2\underline{v_2}$

$= c_1\underline{v_1} + c_2 \cdot (-\tfrac{1}{2}\underline{v_1})$

$= (c_1 - \tfrac{1}{2}c_2)\underline{v_1}$

Linear independence

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ (n-vectors)

Defn: The vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}\}$ are called linearly dependent if at least one vector is a linear combination of the others, and linearly independent otherwise.

Ex: $\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ such that $\Rightarrow \underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}$ linearly dependent and

$\underline{v_4} = \underline{v_1} - 2\underline{v_2} + \underline{v_3}$ $\quad \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4}) = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$

↑ lin. dependency rel.

$$\boxed{\underline{v_1} - 2\underline{v_2} + \underline{v_3} - \underline{v_4} = \underline{0}}$$

Linear system with augmented matrix

$$\begin{array}{ccccc} \left(\underline{v_1}\right. & \underline{v_2} & \cdots & \underline{v_r} & \left.\underline{0}\right) \\ x_1 & x_2 & & x_r & \end{array}$$

← homogeneous

non-trivial soln's

only $\underline{x} = \underline{0}$ solution

Method: Write down the vector eqn. $x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + \ldots + x_r \cdot \underline{v_r} = \underline{0}$, and solve it.

Result: $\underline{v}_1, \dots, \underline{v}_r$ (n-vectors)

Consider the vector eqn. $x_1\underline{v}_1 + \dots + x_r\underline{v}_r = \underline{0}$. We have:

i) One solution $\underline{x} = \underline{0}$: $\underline{v}_1, \dots, \underline{v}_r$ linearly independent

ii) Non-trivial soln's $\underline{x} \neq \underline{0}$: — " — linearly dependent

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\underline{v}_4 = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$x_1\underline{v}_1 + x_2\underline{v}_2 + x_3\underline{v}_3 + x_4\underline{v}_4 = \underline{0}$

$$\begin{pmatrix} 1 & 3 & 1 & 5 \\ 2 & -1 & 1 & 2 \\ 4 & 0 & -1 & 3 \end{pmatrix} \begin{matrix} \\ -2 \\ -4 \end{matrix} \to \begin{pmatrix} 1 & 3 & 1 & 5 \\ 0 & -7 & -1 & -8 \\ 0 & -12 & -5 & -17 \end{pmatrix} \begin{matrix} \\ \\ -12/7 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 3 & 1 & 5 & 0 \\ 0 & -7 & -1 & -8 & 0 \\ 0 & 0 & -23/7 & * & 0 \end{array}\right)$$

i) $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ linearly dependent $\Leftarrow$ one free var. $\Rightarrow$ non-trivial soln's

ii) $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ lin independent and $\underline{v}_4$ is a lin. comb. of $\underline{v}_1, \underline{v}_2, \underline{v}_3$

x_1, x_2, x_3 basic

x_4 free

$\underline{x_4 = 1}$: one possible sol. for x_1, x_2, x_3

$\downarrow$

Non-trivial soln: $(x_1, x_2, x_3, 1)$

$\leftarrow$

$x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + x_3 \cdot \underline{v}_3 + 1 \cdot \underline{v}_4 = \underline{0}$

$\underline{v}_4 = -x_1\underline{v}_1 - x_2\underline{v}_2 - x_3\underline{v}_3$

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Result: The following conditions are equivalent:

i) $\underline{v_1}, \dots, \underline{v_r}$ are linearly independent $\iff$ ii) $x_1\underline{v_1} + \dots + x_r\underline{v_r} = \underline{0}$ has no non-trivial solutions (that is, only the trivial soln) $\iff$ iii) $\text{rk}(\underline{v_1}|\underline{v_2}|\dots|\underline{v_r}) = r$

In particular, there can be at most n linearly independent vectors in $\mathbb{R}^n$.

② Vector spaces, bases and dimensions

Defn: A subset V of $\mathbb{R}^n$ is called a <u>vector space</u> if

i) if $\underline{v}, \underline{w}$ in V then $\underline{v}+\underline{w}$ in V

ii) if $\underline{v}$ in V, r ~~is~~ a number, then $r\underline{v}$ in V

Result: A subset V of $\mathbb{R}^n$ is a vector space if and only if $V = \text{span}(\underline{v_1}, \underline{v_2}, \dots, \underline{v_r})$ for some vectors $\underline{v_1}, \underline{v_2}, \dots, \underline{v_r}$.

Defn: A <u>base</u> of a vector space is a minimal spanning set of vectors. The <u>dimension</u> of a vector space is the number of vectors in a base.

Ex: $V = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4})$, where $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\underline{v_4} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

$= \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$, and $\underline{v_1}, \underline{v_2}, \underline{v_3}$ lin. independent

Hence $B = \{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ is a base of V and $\dim V = 3$

Column space

$A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_r})$: $\text{Col}(A) = \text{span}(\underline{v_1}, \dots, \underline{v_r})$ is a vector space called the column space of A.

Result: i) $\dim \text{Col}(A) = \text{rk}\, A$

ii) The column vectors corresponding to pivot positions is a base of $\text{Col}(A)$.

Explanation. The vectors corresponding to pivots are linearly indep. The remaining vectors are lin. comb. of the first vectors (vectors with pivots).

Ex: $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\text{Col}(A) = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3})$: Base: $\{\underline{v_1}, \underline{v_2}\}$ $\underline{v_3}$ is a lin. comb. of $\underline{v_1}, \underline{v_2}$.

$\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ $v_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

$\dim \text{Col}(A) = \text{rk}\, A = \underline{\underline{2}}$

Null space:

$\text{Null}(A) =$ all solutions of the homog. linear system $(A | \underline{0})$

Result: i) $\text{Null}(A)$ is a vector space

ii) $\text{rk}\, \text{Null}(A) = \underbrace{n - \text{rk}(A)}_{\text{= no. of degrees of freedom}}$, where n is the no. of cols in A.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -1 \\ 1 & 3 & -1 & 3 \end{pmatrix}$

Null(A):

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & -1 & 0 \\ 1 & 3 & -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

x_1, x_2 : basic

x_3, x_4 : free

$$x_1 + x_2 + x_3 + x_4 = 0 \qquad x_1 = -x_4 - x_3 - (x_3 - x_4) = \underline{-2x_3}$$

$$-2x_2 + 2x_3 - 2x_4 = 0 \qquad \frac{-2x_2}{-2} = \frac{-2x_3 + 2x_4}{-2} \qquad \underline{x_2 = x_3 - x_4}$$

$$\underline{x} = \begin{pmatrix} -2x_3 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_4 \\ 0 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= x_3 \cdot \underline{w_1} + x_4 \cdot \underline{w_2}$$

$\text{Null}(A) = \text{span}(\underline{w_1}, \underline{w_2})$

$\dim \text{Null}(A) = \underline{\underline{2}}$ Base $\{\underline{w_1}, \underline{w_2}\}$

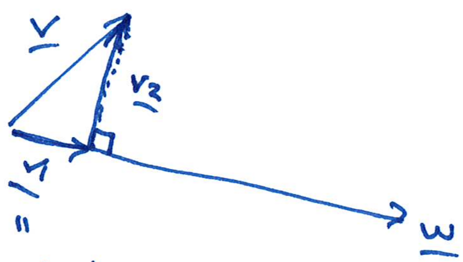
$(n - rk(A) = 4 - 2 = 2)$

Note: x_3, x_4 free $\Rightarrow \underline{w_1}, \underline{w_2}$ lin. independent

(3) Orthogonality and orthogonal projections

<u>Defn:</u> the vectors $\underline{v}, \underline{w}$ are called <u>orthogonal</u> $(\underline{v} \perp \underline{w})$ if $\underline{v} \cdot \underline{w} = 0$. Recall that $\underline{v} \cdot \underline{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$

<u>Orthogonal projections</u>



$\underline{v_1} = \text{proj}_{\underline{w}}(\underline{v}) = \alpha \underline{w}$

the orthogonal projection of $\underline{v}$ onto $\underline{w}$ is the vector $\underline{v_1}$ s.t.

i) $\underline{v} = \underline{v_1} + \underline{v_2}$

ii) $\underline{v_1} = \alpha \underline{w}$

iii) $\underline{v_2} \cdot \underline{w} = 0$

<u>Explanation:</u>

$$\underline{v_2} \cdot \underline{w} = 0$$

$$(\underline{v} - \underline{v_1}) \cdot \underline{w} = 0$$

$$(\underline{v} - \alpha \underline{w}) \cdot \underline{w} = 0$$

$$\underline{v} \cdot \underline{w} - \alpha \underline{w} \cdot \underline{w} = 0$$

$$\underline{v} \cdot \underline{w} = \alpha \cdot \underline{w} \cdot \underline{w}$$

$$\alpha = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}}$$

<u>Formula:</u>

$$\boxed{\text{proj}_{\underline{w}}(\underline{v}) = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \cdot \underline{w}}$$

<u>Ex:</u> $\underline{v} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$\text{proj}_{\underline{w}}(\underline{v}) = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \cdot \underline{w}$$

$$= \frac{2 - 3 + 0}{4 + 1 + 0} \underline{w} = -\frac{1}{5} \underline{w} = \underline{\underline{-\frac{1}{5} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}}$$