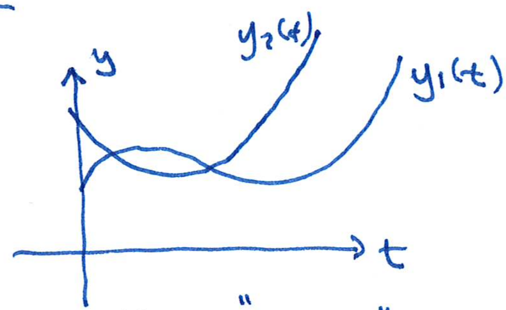
Plan

1 Systems of linear first order differential equations
2 Equilibrium states and stability

Mon: Plenary Session 4

Lecture 11-12 + Exam Problems

① Systems of differential equations



y_1, y_2 "coupled"

Ex: $y_1' = y_1 + 2y_2 - 4$
$y_2' = 2y_1 + y_2 - 5$

$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$: $\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

$\mathbf{y}' = A\mathbf{y} + \mathbf{b} \iff \mathbf{y}' - A\mathbf{y} = \mathbf{b}$

$\boxed{y' + ay = b}$

Defn. A system of differential equations that can be written in matrix form as $\mathbf{y}' = A \cdot \mathbf{y} + \mathbf{b}$ is called systems of linear first order autonomous differential equations. It is called homogeneous if $\mathbf{b} = \mathbf{0}$, and inhomogeneous otherwise.

Note: A is a square matrix

$\mathbf{y}' = A\mathbf{y} + \mathbf{b}$:
$$\boxed{\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n + b_1 \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n + b_2 \\ &\ \ \vdots \qquad\qquad\qquad\qquad \vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n + b_n \end{aligned}}$$

a) The homogeneous case: $\underline{y}' = A\underline{y}$

Ex:

i) $y_1' = -y_1$, $y_2' = 3y_2$

$\underline{y}' = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \underline{y}$

"good" A diagonal
"decoupled"

$y_1' + y_1 = 0$ hom.

$r + 1 = 0 \Rightarrow r = -1$

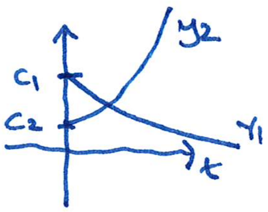
$\Rightarrow y_1 = C_1 \cdot e^{-t}$

$y_2' - 3y_2 = 0$ hom.

$r - 3 = 0 \Rightarrow r = 3$

$\Rightarrow y_2 = C_2 e^{3t}$

$$\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 e^{-t} \\ C_2 e^{3t} \end{pmatrix}$$



$y_1 = C_1 e^{-t}$
$y_2 = C_2 e^{3t}$

$$\underline{y} = C_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + C_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}$$

$\lambda_1 = -1$ $\lambda_2 = 3$

$\underline{v_1}$ $\underline{v_2}$

(eigenvectors)

$$\left.\begin{aligned} z_1 &= -y_1 + y_2 \\ z_2 &= y_1 + y_2 \end{aligned}\right\} \quad \begin{aligned} z_1' &= -1 \cdot z_1 \\ z_2' &= 3 z_2 \end{aligned} \quad \Leftarrow$$

ii) $y_1' = y_1 + 2y_2$, $y_2' = 2y_1 + y_2$

$\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \underline{y}$

"bad" A diagonalizable but not diagonal

Eigenvalues of A:

$\lambda^2 - 2\lambda - 3 = 0$

$(\lambda - 3)(\lambda + 1) = 0$

$\underline{\lambda_1 = -1}$, $\underline{\lambda_2 = 3}$

Eigenvectors of A:

Base:

E_{-1}: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $2x + 2y = 0$, y free, $x = -y$ $\rightarrow \underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

E_3: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ $-2x + 2y = 0$, y free, $x = y$ $\rightarrow \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution:

$$\underline{y} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t}$$
$$= C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$y_1 = C_1 \cdot (-1) e^{-t} + C_2 e^{3t}$
$y_2 = C_1 e^{-t} + C_2 e^{3t}$

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$: $-y_1 + y_2$ $\quad (-y_1 + y_2)' = y_1 - y_2 = -1(-y_1 + y_2)$

$\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $y_1 + y_2$ $\quad (y_1 + y_2)' = 3(y_1 + y_2)$

iii) $y_1' = y_1 + 2y_2$, $y_2' = y_2$

$\underline{y}' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \underline{y}$

A not diagonal.

Formula:

If $\underline{y}' = A\underline{y}$ is homogeneous and A is an $n \times n$-matrix that is diagonalizable, then the general solution is

$$\underline{y} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t} + \dots + C_n \underline{v_n} e^{\lambda_n t}$$

Assume: $\underline{y}' = A\underline{y}$

$$P^{-1}AP = D \quad \Updownarrow \quad AP = PD$$

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad P = \left(\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n} \right)$$

Change of variables:

$$\underline{y} = P \cdot \underline{z} \iff \underline{z} = P^{-1} \cdot \underline{y}$$

L.H.S: $\underline{y}' = (P\underline{z})' = P \cdot \underline{z}'$

R.H.S: $A\underline{y} = AP\underline{z} = PD\underline{z}$

$$\Downarrow$$

$$P^{-1} \cdot \mid \quad P\underline{z}' = PD\underline{z}$$

$$\underline{z}' = D\underline{z}$$

$$z_1' = \lambda_1 z_1 \qquad z_1' - \lambda_1 z_1 = 0$$

$$z_2' = \lambda_2 z_2$$

$$\vdots$$

$$z_n' = \lambda_n z_n$$

$$z_1 = C_1 e^{\lambda_1 t}$$

$$z_2 = C_2 \cdot e^{\lambda_2 t}$$

$$\vdots$$

$$z_n = C_n \cdot e^{\lambda_n t}$$

$$\underline{z} = \begin{pmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \\ \vdots \\ C_n e^{\lambda_n t} \end{pmatrix} = C_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_2 t} + \dots$$

$$\underline{y} = P \cdot \left(C_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_1 t} + C_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_2 t} + \dots \right)$$

$$= C_1 \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t} + \dots + C_n \underline{v_n} e^{\lambda_n t}$$

$$\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \underline{y}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$z_1 = \frac{1}{2}(-y_1 + y_2)$$

$$z_2 = \frac{1}{2}(y_1 + y_2)$$

ii) <u>The inhomogeneous case</u>: $\underline{y}' = A\underline{y} + \underline{b}$

<u>Ex</u>: $y_1' = y_1 + 2y_2 - 4$
$y_2' = 2y_1 + y_2 - 5$
$\underline{y}' = A\underline{y} + \underline{b}$ $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

$\boxed{\text{Superposition principle: } y = y_h + \underline{y_p}, \text{ where: } y_h: \text{gen. solution of } \underline{y}' = A\underline{y};\ y_p: \text{part. sol. of } \underline{y}' = A\underline{y} + \underline{b}}$

<u>Ex</u>: $\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\underline{y} + \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

$$\underline{y} = y_h + y_p = \underline{\underline{C_1\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}}}$$

y_h: $\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\underline{y} \Rightarrow y_h = \underline{C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t} + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t}}$

y_p: $\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\underline{y} + \begin{pmatrix} -4 \\ -5 \end{pmatrix} \Leftrightarrow \underline{y}' - \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\underline{y} = \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

$\left.\begin{array}{l} \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ const. sol.} \quad y_1 = A \\ \underline{y}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad y_2 = B \end{array}\right\}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} -4 \\ -5 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$y_p = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \end{array}\right) \rfloor -2$

$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array}\right)$

$A + 2B = 4$
$-3B = -3$
$\underline{B = 1}$ $\underline{A = 2}$

<u>In general</u>:

$A \cdot y_p = -\underline{b}$

② Equilibrium states and stability:

$\underline{y}' = A\underline{y} + \underline{b}$ system of diff. eqns.

Equilibrium state = constant solutions $\underline{y}' = \underline{0}$

$A\underline{y} + \underline{b} = \underline{0}$

Note: Eq. states = particular solutions $A\underline{y} = -\underline{b}$

$y_e = y_p$

Stability of an eq. state y_e:

$\underline{y_e}$ is stable if $\underline{y}(0)$ close to $\underline{y_e}$ implies that $y(t) \to \underline{y_e}$ as $t \to \infty$

$$\underline{y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{pmatrix}$$

$\underline{y_e}$ is unstable otherwise

Ex: $\underline{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \underline{y} + \begin{pmatrix} -4 \\ -5 \end{pmatrix}$ $\quad y_e = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

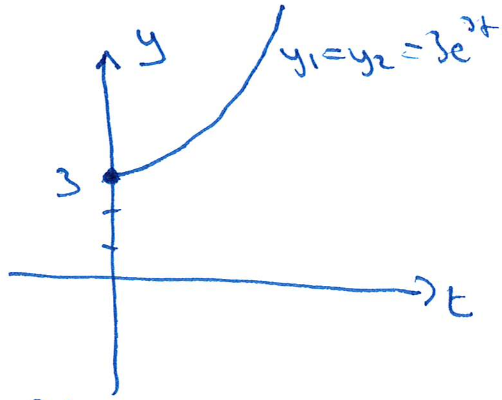
$$\underline{y} = C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lim_{t\to\infty} y(t)$: $\quad e^{-t} \to 0$, $\quad e^{3t} \to \infty$

Result: An eq. state y_e is stable $\Leftrightarrow$ A negative defn.
(and gl. asympt. stable)

<u>Ex:</u> $\underbrace{y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} y}$, $y(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$y_1(0) = 3$
$y_2(0) = 3$



$$y = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$y_1 = -c_1 e^{-t} + c_2 e^{3t}$$
$$y_2 = c_1 e^{-t} + c_2 e^{3t}$$

$y(0) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$: $\quad c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^0 + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^0 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$$c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 1 & 3 \\ 1 & 1 & 3 \end{array}\right) \qquad \left(\begin{array}{cc|c} -1 & 1 & 3 \\ 0 & 2 & 6 \end{array}\right)$$

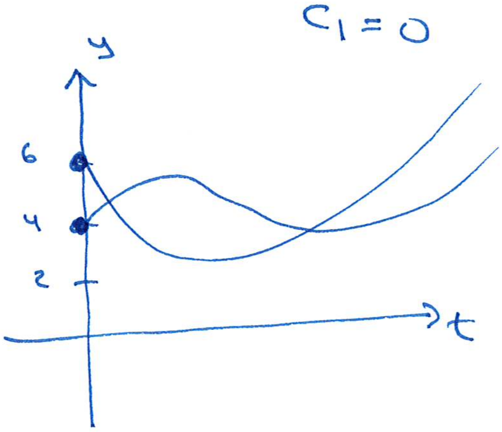
$-c_1 + 3 = 3$
$c_2 = 3$
$c_1 = 0$

<u>Particular solution:</u>

$$y = 0 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$y_1 = 3e^{3t}$$
$$y_2 = 3e^{3t}$$

<u>Ex:</u> $y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} y$, $y(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$



$$y = c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$y(0) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$: $\quad c_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$$\left(\begin{array}{cc|c} -1 & 1 & 4 \\ 1 & 1 & 6 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 4 \\ 0 & 2 & 10 \end{array}\right) \qquad \begin{array}{l} -c_1 + 5 = 4 \\ c_2 = 5 \\ c_1 = 1 \end{array}$$

$$y = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \qquad \begin{array}{l} y_1 = -e^{-t} + 5e^{3t} \\ y_2 = e^{-t} + 5e^{3t} \end{array}$$

Relationship with second order differential equations

Ex: $y'' - 7y' + 12y = 3$

$Y = Y_h + Y_p = \underline{\underline{C_1 e^{3t} + C_2 e^{4t} + 1/4}}$ ($e^{3t} \to \infty$, $e^{4t} \to \infty$)

$\underline{Y_h}$: $y'' - 7y' + 12y = 0$

$r^2 - 7r + 12 = 0$

$(r-3)(r-4) = 0$

$r = 3,\ r = 4$

$Y_h = \underline{C_1 e^{3t} + C_2 e^{4t}}$

$\underline{y_p}$: $\left.\begin{array}{l} y = A \\ y' = 0 \\ y'' = 0 \end{array}\right\}$ $12A = 3$, $A = 3/12 = 1/4$, $y_p = \underline{1/4}$

$\underline{\text{Eq. } \cancel{\text{state}}}$: $y' = 0,\ y'' = 0$

$\left.\begin{array}{l} y = A \\ y' = 0 \\ y'' = 0 \end{array}\right\}$ $12A = 3$, $A = 3/12 = 1/4$, $y_e = \underline{\underline{1/4}}$

$\underline{\text{Stable}}$ $\Leftrightarrow$ all char roots are negative

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$\underline{\text{Ex:}}$ $y'' - 7y' + 12y = 3$

$\Updownarrow$

$y_1 = y$, $y_2 = y'$

$y_1' = y' = y_2$

$y_2' = y'' = 7y' - 12y + 3$

$= 7y_2 - 12y_1 + 3$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$\underline{y}' = A\underline{y} + \underline{b}$ with $A = \begin{pmatrix} 0 & 1 \\ -12 & 7 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ $\underline{y} = \begin{pmatrix} y \\ y' \end{pmatrix}$

$\underline{\underline{y} = \underline{y}_h + \underline{y}_p} = \underline{\underline{C_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}}}$

$y = C_1 e^{3t} + C_2 e^{4t} + 1/4$

$y' = 3C_1 e^{3t} + 4C_2 e^{4t}$

$\underline{Y_p}$: $A \cdot \underline{y}_e = -\underline{b}$

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -12 & 7 & -3 \end{array}\right) \sim \left(\begin{array}{cc|c} -12 & 7 & -3 \\ 0 & 1 & 0 \end{array}\right)$$

$-12y_1 = -3$, $y_1 = -3/-12 = 1/4$

$y_2 = 0$

$\underline{Y_h}$: $\lambda^2 - 7\lambda + 12 = 0$

$\lambda = 3, \lambda = 4$

$\underline{E_3}$: $\begin{pmatrix} -3 & 1 \\ \cancel{-12} & \cancel{4} \end{pmatrix}$ $\underline{v_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\underline{E_4}$: $\begin{pmatrix} -4 & 1 \\ \cancel{-12} & \cancel{3} \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$