	Plan
1	Equilibrium states and stability
2	Linear second order differential equations

① Equilibrium states and stability

First order diff. eqn: $y' = F(t,y)$

Lecture 10:
- seperable: $y' = f(t)\cdot g(y)$
- linear: $y' + a(t)y = b(t)$
- exact

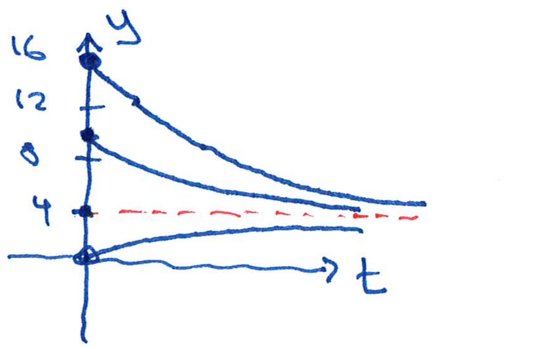
Defn. A first order differential equation is called autonomous if it can be writen $y' = F(y)$. In that case, an equilibrium state is a value $y = y_e$ such that $F(y_e) = 0$. ← $y' = 0$

Ex: $y' = (4-y)\cdot k$

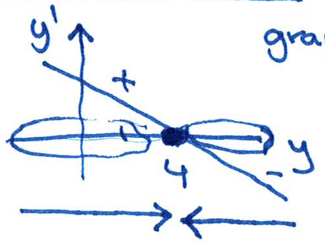
$y' = k(4-y)$ ← Autonomous

$F(y) = 0$

$k\cdot(4-y) = 0 \Rightarrow y_e = 4$



Phase diagram:



graph of $y' = F(y)$

$y' = k(4-y)$

$k(4-y)$ —0····
4

$y_0 = 4$: $y(t) = 4$ const. sol.

$y_0 > 4$: $y(t) \to 4$ as $t \to \infty$

$y_0 < 4$: $y(t) \to 4$ — " —

Defn. An equilibrium state $y = y_e$ for the differential eqn. $y' = F(y)$ is called <u>stable</u> if y_0 close to y_e implies that $y(t) \to y_e$ as $t \to \infty$, and it is called <u>unstable</u> otherwise.

stable eq. state | unstable eq. state | Stable eq. state that is <u>not</u> globally <u>asymptotically</u> <u>stable</u>

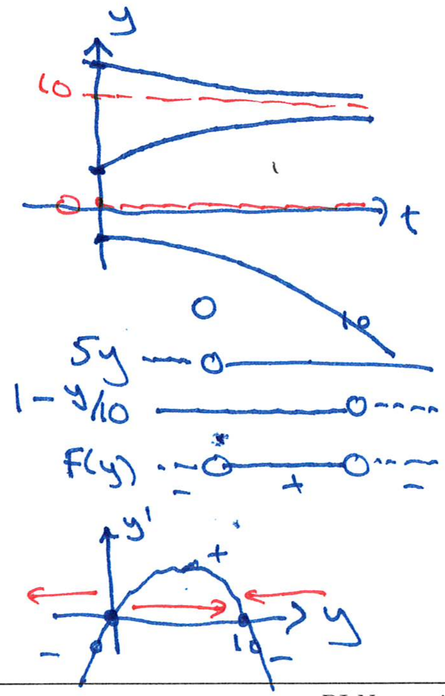
Defn. The eq. state is called <u>globally asymptotically</u> <u>stable</u> if $y(t) \to y_e$ as $t \to \infty$ for <u>any</u> starting state y_0.

Ex: $y' = 5y(1 - y/10)$

Eq. states: $5y(1 - y/10) = 0$

$\underline{y_e = 0}$ and $\underline{y_e = 10}$ $\quad$ $\underline{y=0}$ or $\underline{y=10}$

$$\boxed{\begin{array}{l} y_e = 0 \text{ is \underline{unstable} eq. state} \\ y_e = 10 \text{ is \underline{stable} } -\text{''}- \\ \quad (\text{but \underline{not} gl. asymp. stable}) \end{array}}$$



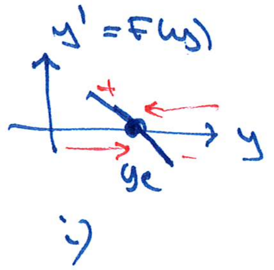
Stability Thm

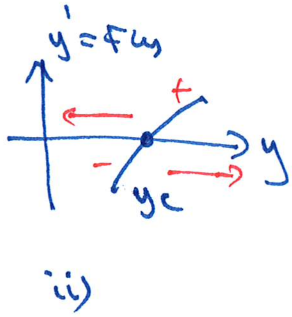
If $y = y_e$ is an eq. state of $y' = F(y)$, then:

i) y_e is stable if $F'(y_e) < 0$

ii) y_e is unstable if $F'(y_e) > 0$

Explanation:





(2) Linear second order differential equations

Ex: $y'' = 6t - 12 \quad | \int \ldots dt$

$$y' = \int 6t - 12\,dt = 6\frac{t^2}{2} - 12t + C = \underline{3t^2 - 12t + C}$$

$$y = \int 3t^2 - 12t + C\,dt = 3 \cdot \frac{t^3}{3} - 12\frac{t^2}{2} + Ct + D$$

$$y = \underline{\underline{t^3 - 6t^2 + Ct + D}}$$ general solution

note: second order diff. equ $\Rightarrow$ two undetermined coeff. in the general solution

Defn: A second order differential equation is called linear with constant coefficients if it can be written

$$\boxed{y'' + a \cdot y' + by = c(t)} \quad \begin{cases} a, b \text{ constants} \\ c(t) \text{ expr. in } t \end{cases}$$

It is called homogeneous if $c(t) = 0$; and inhomogeneous otherwise

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(a) the homogeneous case: $y''+ay'+by=0$

Ex: $y''-4y'+3y=0$

Char. eqn: $r^2-4r+3=0 \Rightarrow y=C_1e^{1\cdot t}+C_2\cdot e^{3\cdot t}$

$(r-1)(r-3)=0 \qquad = C_1e^{t}+C_2e^{3t}$

$y''\to r^2$

$y'\to r$

$y\mapsto 1$

$r=1$, $r=3$

(char. roots).

e^t, e^{3t} are sol's

Explanations: $y''+ay'+by=0$

Look for solutions $y=e^{rt}$

$y'=e^{rt}\cdot r=re^{rt}$

$y''=r(e^{rt}\cdot r)=r^2e^{rt}$

$$\left.\begin{array}{l} y''+ay'+by=0 \\ r^2e^{rt}+a(re^{rt}) \\ \quad +b\cdot(e^{rt})=0 \\ e^{rt}(r^2+ar+b)=0 \\ r^2+ar+b=0 \end{array}\right.$$

Concl: $y=e^{rt}$ is a solution $\Leftrightarrow r^2+ar+b=0$

Compare this with: $A\cdot\underline{x}=\underline{0}$ lin. homog. system. Solutions: span$(\underline{w}_1,\dots,\underline{w}_r)$, $r=\#$ degrees of freedom

Ex: $y''-4y'+4y=0$

$y''-4y'+4y=0$

Char. eqn: $r^2-4r+4=0$

$(r-2)^2=0$

$r_1=2 \quad r_2=2$

Sol's: $y=e^{2t}$, $y=e^{2t}$

$\Rightarrow y=C_1\cdot e^{2t}+C_2\cdot te^{2t}$

Note: if r is a double char. root, then e^{rt} and $t\cdot e^{rt}$ are sol's of the diff. eqn.

Ex: $y''-4y'+5y=0$

Char. eqn: $r^2-4r+5=0$

$r=\dfrac{4\pm\sqrt{16-4\cdot1\cdot5}}{2\cdot1}$

$=2\pm\frac{1}{2}\cdot\sqrt{-4}$

$=2\pm\frac{1}{2}\cdot\sqrt{4}\cdot\sqrt{-1}=2\pm\sqrt{-1}$

$\Rightarrow y=e^{2t}(C_1\cos t+C_2\sin t)$

(b) The inhomogeneous case $y''+ay'+by=c(t)$

Ex: $y''-4y'+3y=e^{2t} \rightsquigarrow y=y_h+y_p$

Superposition principle:

The general solution of $y''+ay'+by=c(t)$ is given by $y=y_h+y_p$, where

- y_h: the general solution of the homogeneous eqn. $y''+ay'+by=0$
- y_p: a particular solution of the inhomogeneous eqn. $y''+ay'+by=c(t)$

Ex: $y''-4y'+3y=e^{2t}$ $\quad y=y_h+y_p=C_1e^t+C_2e^{3t}+\ldots$

y_h: $y''-4y'+3y=0$

Char. eqn: $r^2-4r+3=0 \Rightarrow y_h=C_1e^t+C_2e^{3t}$

$r=1,\ r=3$

y_p: $y''-4y'+3y=e^{2t}$

$y=Ae^{2t}$ or $\cancel{y=e^{at}}$

$c(t)=e^{2t}$

$c'(t)=2e^{2t}$

$c''(t)=4e^{2t}$

Method of undetermined coeff:

choose an expr. of y such that

- y has "the same form" as $c(t)$
- y has some undetermined coeff.
- Compute $c'(t), c''(t)$ and choose y of the same form as c, c', c''

$$\left.\begin{aligned} y &= \underline{Ae^{2t}} \\ y' &= A\cdot e^{2t}\cdot 2 = 2Ae^{2t} \\ y'' &= 2A(e^{2t}\cdot 2) = 4Ae^{2t} \end{aligned}\right\}$$

$$y'' - 4y' + 3y = e^{2t}$$

$$4Ae^{2t} - 4(2Ae^{2t}) + 3(Ae^{2t}) = e^{2t}$$

$$e^{2t}\cdot\underbrace{(4A - 8A + 3A)}_{-A} = e^{2t}$$

$$-A = 1$$

$$A = -1 \qquad \underline{y_p = -e^{2t}}$$

$$y = y_h + y_p = \underbrace{C_1e^t + C_2e^{3t}}_{y_h} \underbrace{- e^{2t}}_{y_p} = \underline{\underline{C_1e^t + C_2e^{3t} - e^{2t}}}$$

Lin. sys. (not homog.)

$A\underline{x} = \underline{b}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 1 & 5 & 9 \\ 4 & 2 & 6 & 12 \end{array}\right) \begin{array}{l} \\ -3 \\ -4 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 2 & 0 \\ \cancel{0} & \cancel{-2} & \cancel{2} & \cancel{0} \end{array}\right)$$

$$\left\{\begin{aligned} x + y + z &= 3 \qquad \underline{x = 3 - 2z} \\ -2y + 2z &= 0 \qquad \underline{y = z} \\ z &\text{ free} \end{aligned}\right.$$

$$\underline{x} = \begin{pmatrix} 3-2z \\ z \\ z \end{pmatrix} = \underbrace{z\cdot\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{\underline{x}_h} + \underbrace{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}}_{\underline{x}_p}$$

Ex: $y'' - 7y' + 12y = t$

$$y = y_h + y_p = \underline{\underline{C_1 e^{3t} + C_2 e^{4t} + \frac{1}{12}t + \frac{7}{144}}}$$

y_h: $y'' - 7y' + 12y = 0$

$r^2 - 7r + 12 = 0$

$(r-3)(r-4) = 0$

$\underline{r=3}, \underline{r=4} \Rightarrow y_h = \underline{C_1 e^{3t} + C_2 e^{4t}}$

y_p: $y'' - 7y' + 12y = t$

$$\begin{array}{l} c(t) = t \\ c' = 1 \\ c'' = 0 \end{array} \qquad \begin{array}{l} y = At + B \\ y' = A \\ y'' = 0 \end{array}$$

$0 - 7 \cdot A + 12(At + B) = t$

$(12A)t + (-7A + 12B) = 1 \cdot t + 0$

$12A = 1 \qquad A = 1/12$

$-7A + 12B = 0 \qquad -7/12 + 12B = 0 \qquad B = 7/144$

$$\Rightarrow y_p = \underline{\frac{1}{12}t + \frac{7}{144}}$$

Ex: $y'' - 3y' + 2y = 4e^{2t}$ $\quad y = y_h + y_p = \underline{C_1 e^{t} + C_2 e^{2t} + 4te^{2t}}$

y_h: $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1,\ r = 2$ $\quad \Rightarrow y_h = \underline{C_1 e^{t} + C_2 e^{2t}}$

y_p: $c(t) = 4e^{2t}$ $\quad y'' - 3y' + 2y = 4e^{2t}$

$$\left.\begin{aligned} y &= Ae^{2t} \\ y' &= 2Ae^{2t} \\ y'' &= 4Ae^{2t} \end{aligned}\right\} \quad \begin{aligned} &4Ae^{2t} - 3(2Ae^{2t}) + 2(Ae^{2t}) = 4e^{2t} \\ &e^{2t}(\underbrace{4A - 6A + 2A}_{0 \cdot A \,=\, 4}) = 4e^{2t} \end{aligned}$$

Try: $\boxed{y = Ate^{2t}}$ $\quad \Rightarrow (4A + 4At)e^{2t}$

$\qquad - 3(A + 2At)e^{2t}$

$\qquad + 2(At)e^{2t} = 4e^{2t}$

$y' = A \cdot e^{2t} + At \cdot (2e^{2t})$

$= \underline{(A + 2At)e^{2t}}$

$y'' = 2A \cdot e^{2t} + (A + 2At) \cdot e^{2t} \cdot 2$ $\quad (\cancel{4At - 6At + 2At}$

$= \underline{(4A + 4At)e^{2t}}$ $\quad + 4A - 3A\ )e^{2t} = 4e^{2t}$

$A = 4$ ok

$y_p = \underline{4te^{2t}}$

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