Plan

1 Separable differential equations
2 Linear first order differential equations
3 Exact differential equations

Plenary Session 3:
Thu at 16-19 in ~~A1-040~~
A1-040
Lecture 8-10
+ Exam Questions

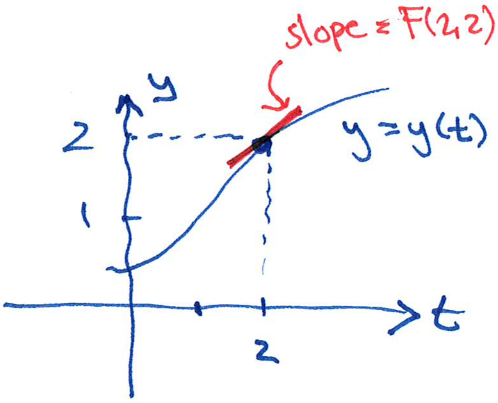
Introduction to differential equations

- the unknown is a function
- the equations contain derivatives of the unknown function

ODE (ordinary differential eqn.)
= function in one variable

Independent variable: t = time
Dependent variable: $y = y(t)$
Growth: $y' = y'(t)$
(marginal growth per time unit)

Order: the highest order derivative in the diff. eqn.



$$\boxed{y' = F(t,y)}$$
First order diff. eqn.

At $(2,2)$: $y' = F(2,2)$
(t,y)

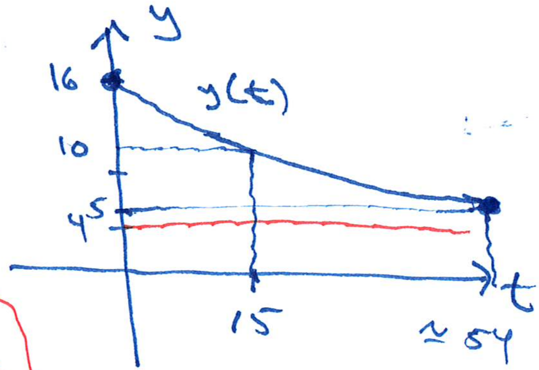
① Separable differential equations

Ex: A coca-cola can has room temp. 16°C and you put it in a fridge with const. temp 4°C. Let $y(t)$ be the temp. of the can after t mins.

$y' = k(4-y)$ k const. (positive), $y(0) = 16$

$$y' = k\cdot(4-y) \qquad |:(4-y)$$

$$\frac{1}{4-y}\cdot y' = k \qquad \left| \int \dots dt \right.$$



$$\int \frac{1}{4-y} y'\,dt = \int k\,dt$$

$u = 4-y$
$du = u'(t)\,dt$
$du = -y'\,dt \Rightarrow y'\,dt = \frac{du}{-1}$

$$\int \frac{1}{u}\frac{du}{-1} = kt + C_2$$

$$\int -\frac{1}{u}\,du = kt + C_2$$

$$= -\ln|u| + C_1 = kt + C_2$$

$$-\ln|4-y| + C_1 = kt + C_2$$ general sol'n in implicit form

$$-\ln|4-y| = kt + C_2 - C_1$$

$$\ln|4-y| = -kt - C_2 + C_1 \qquad |e^{\cdot}$$

$$|4-y| = e^{-kt-C_2+C_1} = e^{-kt}\cdot e^{C_1-C_2}$$

$$4-y = \pm e^{C_1-C_2}\cdot e^{-kt} = C\cdot e^{-kt}$$

$$y = \underline{\underline{4 - C\cdot e^{-kt}}}$$ general solution (explicit form)

$\underline{y(0)=16}$: Initial condition

$$4 - C\cdot e^{-k\cdot 0} = 16$$

$$4 - C = 16 \Rightarrow \underline{C = -12}$$

$$Y = \underline{\underline{4 + 12e^{-kt}}}$$ particular solution

Assume:

Temp. of the can after 15 min is 10°C.

$y(15) = 10$:

$$4 + 12e^{-k\cdot 15} = 10$$

$$12e^{-15k} = 10-4$$

$$e^{-15k} = \frac{10-4}{12} = 1/2$$

$$-15k = \ln(1/2)$$

$$-k = \frac{1}{15}\ln(1/2)$$

$$-k = \ln(1/2)^{1/15}$$

$$y = 4 + 12(e^{-k})^t$$

$$= 4 + 12(1/2)^{1/15\,t}$$

$$y = \underline{\underline{4 + 12\cdot(1/2)^{t/15}}}$$

When is $y(t)=5$?

$$4 + 12\cdot(1/2)^{t/15} = 5 \Rightarrow (1/2)^{t/15} = \frac{5-4}{12} = \frac{1}{12} \Rightarrow \frac{t}{15}\ln(1/2) = \ln(1/12)$$

$$\frac{t}{18} = \frac{\ln(1/12)}{\ln(1/2)} = \frac{-1\cdot\ln(12)}{-1\cdot\ln(2)} = \frac{\ln 12}{\ln 2} \Rightarrow t = 15\cdot\frac{\ln 12}{\ln 2}$$
$$\approx \underline{\underline{54 \text{ min.}}}$$

Defn: A first order differential equation is separable if it can be written $y' = f(t)\cdot g(y)$

Ex:

$y' = k\cdot(4-y)$ is separable $\quad f(t)=k \quad g(y)=(4-y)$

$y' = t+y$ is not separable

$y' = 0.15y$ is separable $\quad f(t)=0.15 \quad g(y)=y$

$y' = 2t^2$ — " — $\quad f(t)=2t^2 \quad g(y)=1$

Solution method:

$y' = f(t)\cdot g(y) \quad | : g(y)$

$\frac{1}{g(y)}y' = f(t) \quad | \int \ldots dt$

$\int \frac{1}{g(y)} y'\,dt = \int f(t)\,dt$

$\boxed{u = y \\ du = y'\,dt}$

$\int \frac{1}{g(u)}\,du = \int f(t)\,dt$

$$\boxed{\int \frac{1}{g(y)}\,dy = \int f(t)\,dt}$$

gives general solution in implicit form

Ex:

$y' = 0.15y \quad | : y$

$\frac{1}{y}y' = 0.15$

$\int \frac{1}{y}y'\,dt = \int 0.15\,dt$

$\int \frac{1}{y}\,dy = \int 0.15\,dt$

$\ln|y| + c_1 = 0.15t + c_2$

$\ln|y| = 0.15t + c_2 - c_1$

$|y| = e^{0.15t}\cdot e^{c_2-c_1}$

$y = \pm e^{c_2-c_1}\cdot e^{0.15t}$

gen. sol. $\quad y = \underline{\underline{C\cdot e^{0.15t}}}$

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Note: Any first order differential equation has a general solution that depends on one undetermined coeff. You need one initial condition to determine this coeff.

Special case: $y' = f(t) \Rightarrow y = \int f(t)\,dt$

Ex: $y' = 3t^2 \Rightarrow y = \int 3t^2\,dt = t^3 + C$ $\quad y = t^3 + C$ general sol'n.

2 Linear differential equations

What about $y' = y + t$?

Defn. A first order differential equ. is linear if it can be written $\boxed{y' + a(t)y = b(t)} \Leftrightarrow y' = \underbrace{b(t) - a(t)y}_{\text{lin. expr. in } y}$

Ex: $y' = y + t$ lin.

$y' - y = t \quad \begin{cases} a(t) = -1 \\ b(t) = t \end{cases}$

Solution method for $y' + a(t)y = b(t)$: Integrating factor

$y' + ay = b \quad | \cdot u$ $\quad a = a(t)$, $b = b(t)$, $y = y(t)$

$uy' + auy = bu$

$uy' + u'y = bu$

$(u \cdot y)' = bu$

$uy = \int ub\,dt \Rightarrow y = \frac{1}{u(t)} \cdot \int u(t)b(t)\,dt$

Requirement: $u' = a \cdot u$

integrating factor $\boxed{u = e^{\int a(t)dt}}$

$u' = \left(e^{\int a(t)dt}\right)' = e^{\int a(t)dt} \cdot a(t) = u \cdot a(t)$

Ex: $y' = y + t$

$y' - y = t \quad | \cdot e^{-t}$

$(y \cdot e^{-t})' = te^{-t}$

$y \cdot e^{-t} = \int te^{-t}\,dt \quad | \cdot e^{t}$

$y = e^{t} \cdot \int te^{-t}\,dt$

$= e^{t}(-te^{-t} - e^{-t} + C)$

$y = \underline{\underline{-t - 1 + Ce^{t}}}$

general sol'n.

Int. factor: $\underline{u = e^{-t}}$

$a(t) = -1$

$\int a(t)\,dt = \int -1\,dt$
$= -t + C$

$u' = e^{-t} \cdot (-1) = u \cdot a$

$\int te^{-t}\,dt =$

$\boxed{\begin{array}{ll} u = -e^{-t} & v = t \\ u' = e^{-t} & v' = 1 \end{array}}$

Int. by parts:

$\int u'v\,dt = uv - \int uv'\,dt$

$= -e^{-t} \cdot t - \int -e^{-t} \cdot 1\,dt$

$= -te^{-t} + \int e^{-t}\,dt$

$= \underline{-te^{-t} - e^{-t} + C}$

Alternative method: for linear diff. eqn. where $a(t)$ is const.

Superposition

$$y = y_h + y_p \quad \begin{cases} y_h: \text{ general solution of } y' + ay = 0 \\ y_p: \text{ particular solution of } y' + ay = b(t) \end{cases}$$

Ex: $y' - y = t \qquad y = y_h + y_p = Ce^{t} + (-t - 1) = \underline{\underline{-t - 1 + Ce^{t}}}$

y_h: $y' - y = 0$

char. eqn: $r - 1 = 0 \quad \underline{r = 1} \to y_h = C \cdot e^{1 \cdot t} = Ce^{t}$

$-c = 1 \quad c = -1$
$c - d = 0 \quad d = -1$

y_p: $y' - y = t \qquad \left.\begin{array}{l} y = ct + d \\ y' = c \end{array}\right\} \quad \begin{array}{l} c - (ct + d) = t \\ -ct + (c - d) = t \end{array}$

$y_p = -t - 1$

3) Exact differential equations

Any first order differential equation $y' = F(t,y)$ can always be written $\underline{p(t,y) + q(t,y)\cdot y' = 0}$.

Defn. A first order diff. eqn. is exact if you can write it $\boxed{p(t,y) + q(t,y)\cdot y' = 0}$ such that

$$p(t,y) = h'_t \quad \text{and} \quad q(t,y) = h'_y$$

for some fn. $h(t,y)$.

Solution method: $h(t,y) = C$

Ex: $y' = \dfrac{ye^t + 2e^{2t}}{2y - e^t} \quad | \cdot (2y - e^t)$

$$(2y - e^t)\cdot y' = ye^t + 2e^{2t}$$

$$ye^t + 2e^{2t} - (2y - e^t)y' = 0$$

$$\underbrace{(ye^t + 2e^{2t})}_{p(t,y)} + \underbrace{(e^t - 2y)}_{q(t,y)}y' = 0$$

1 $h'_t = ye^t + 2e^{2t}$

2 $h'_y = e^t - 2y$

2: $h'_y = e^t - 2y$ $\quad h = \int e^t - 2y \, dy$

$$h = \underline{ye^t - y^2 + C(t)}$$

1 $h'_t = (ye^t - y^2 + C(t))'_t = \cancel{y \cdot e^t} - \cancel{0} + C'(t)$

$= \cancel{ye^t} + 2e^{2t}$

$\Rightarrow C'(t) = 2e^{2t} \Rightarrow C(t) = \int 2e^{2t}\,dt = e^{2t} + C$

Concl.: $h = ye^t - y^2 + e^{2t} + C$ yes, the diff.-eqn. is exact

General solution: $h(t,y) = C$

$ye^t - y^2 + e^{2t} + C_1 = C_2$

$\Rightarrow ye^t - y^2 + e^{2t} = C$

Explicit solution: $-y^2 + e^t \cdot y + (e^{2t} - C) = 0$ $a = -1$, $b = e^t$, $c = e^{2t} - C$

$$y = \frac{-e^t \pm \sqrt{e^{2t} - 4(-1)(e^{2t} - C)}}{2 \cdot (-1)}$$

$$= \frac{e^t}{2} \pm \frac{\sqrt{5e^{2t} - 4C}}{2}$$

If exact: $p(t,y) + q(t,y)y' = 0$

$$h'_t(t,y) + h'_y(t,y) \cdot y' = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt} = 0$$

$$\frac{d}{dt} h(t,y) = 0$$

$$h(t,y) = C$$