Plan	
1	Linear systems and their geometry
2	Gaussian elimination
3	Rank of a matrix
4	Homogeneous linear systems

Problem:

① Linear systems and their geometry

Ex:

$$\begin{aligned} x+y+z+w &= 5 \\ 2x-3y+z-w &= 0 \\ x+6y+2z+4w &= 17 \end{aligned}$$

3×4 linear system

Defn:

Linear eqn. in n variables: $\boxed{a_1\cdot x_1 + a_2\cdot x_2 + \dots + a_n\cdot x_n = b}$

- degenerate if $a_1 = a_2 = \dots = a_n = 0$
- non-degenerate otherwise

Geometry:

$n=2$, non-degenerate: $ax+by=c$

a line in $\mathbb{R}^2$ (two-dim. coordinate system)

$b\neq 0$: $y = \dfrac{c-ax}{b}$, $y = (c/b) + (-a/b)x$

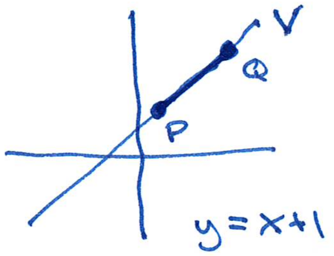
$b=0$, $x = c/a$

$n=3$, — " — : $ax+by+cz=d$ a plane in $\mathbb{R}^3$

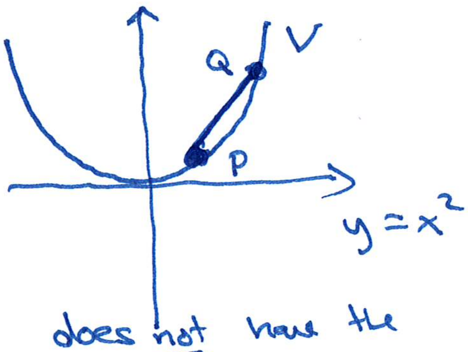
$n>3$, — " — : a hyperplane

Defn. A set V has the <u>linear property</u> if P and Q with $P \neq Q$ in V, then the line segment $[P, Q]$ is in V.

Ex:



has linear property



does <u>not</u> have the linear property

Defn. An $m \times n$ <u>linear system</u> is a system of m linear equ's in n variables. Any linear system has solution set

$$V = \{(x_1, x_2, \ldots, x_n) \mid \text{all linear equ.'s are satisfied}\}$$

Result: The solutions V of a linear system always have the linear property.

Thm: Any $m \times n$ linear system has either

i) no solutions } <u>inconsistent</u>

ii) one unique solution

iii) infinitely many solutions } <u>consistent</u>

Eivind Eriksen (eivind.eriksen@bi.no), Office B3y-085
https://www.dr-eriksen.no/teaching/GRA6035/
https://www.dr-eriksen.no/teaching/ELE3781/
BI Norwegian Business School
MSc in Business/Finance
BSc in Data Science

2 Gaussian elimination: General method for solving any linear system.

Representations of linear systems

$$m \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad \vdots \qquad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n}_{n} = b_m \end{cases}$$

m×n linear system

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad (A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

coeff. matrix (m×n) augmented matrix (extended)

Gaussian process:

a) Elementary row operations

i) Switch two rows
ii) Multiply a row by $c \neq 0$
iii) Add a multiple of one row to another row

Using elementary row operations on the augmented matrix of a linear system will preserve the set of solutions.

b) Echelon form

- The first non-zero element in a row is called a pivot
- A matrix is in echelon form if
  - i) All zero rows are in the bottom of the matrix.
  - ii) Each pivot should be further to the right than the pivots in the rows above.

Ex:

$$x+y+z+w=5$$
$$2x-3y+z-w=0$$
$$x+6y+2z+4w=\cancel{17}\ 15$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -3 & 1 & -1 & 0 \\ 1 & 6 & 2 & 4 & \cancel{17}\ 15 \end{array}\right) \begin{matrix} -2 \\ -1 \\ \end{matrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ x & y & z & w \end{matrix} \end{array}$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 5 & 1 & 3 & \cancel{12}\ 10 \end{array}\right) \begin{matrix} \\ 1 \\ \\ \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -5 & -1 & -3 & -10 \\ 0 & 0 & 0 & 0 & \cancel{2}\ 0 \end{array}\right)$$

echelon form

Defn: Pivot positions = {positions of pivots in an echelon form

(3) Rank of a matrix = number of pivot positions

Facts:
- Any matrix can be transformed into echelon form using elementary row operations
- An echelon form is not unique, but the pivot positions are.

Back substitution:

pivot position in the last col. $\Rightarrow$ inconsistent (no soln's)

$$x+y+z+w=5$$
$$-5y-z-3w=-10$$
$$\cancel{0=2}\ 0$$

$$x = 5-(2-z/5-3w/5)-z-w = 3-4z/5-2w/5$$

$$\frac{-5y}{-5} = \frac{-10+z+3w}{-5} \qquad y = 2-z/5-3w/5$$

x,y: basic var. — pivot pos. in var. col.

z,w: free var. — no pivot position in var. col.

Solutions: $(x,y,z,w) = \left(3 - \frac{4z}{5} - \frac{2w}{5},\ 2 - \frac{z}{5} - \frac{3w}{5},\ z,\ w\right)$

where z, w are free

infinitely many solutions

(two degrees of freedom)

In general: $m \times n$ linear system $(A|\underline{b})$

i) inconsistent (no solutions) $\iff$ pivot pos. in the last col. $\iff \operatorname{rk} A \neq \operatorname{rk}(A|\underline{b})$

ii) if it consistent, then: $\operatorname{rk} A = \operatorname{rk}(A|\underline{b})$

- no free variables $\iff$ one unique solution
- at least one free variable $\iff$ inf. many solutions

$n - \operatorname{rk}(A) =$ # degrees of freedom

Solution set V: $m \times n$ linear system $(A|\underline{b})$

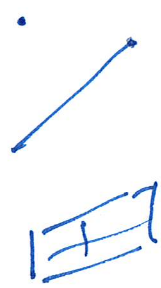
$V =$ the set of all solutions of the linear system

1. V subset of $\mathbb{R}^n$
2. V has the linear property
3. $\dim V = n - \operatorname{rk}(A)$
   $=$ # free variables
   (if the system is consistent)

no free var

one free var.

two free var



(4) Homogeneous linear systems

Defn: An $m\times n$ homogeneous linear system is an $m\times n$ linear system where $\underline{b}=\underline{0}$, i.e.

$$\boxed{\begin{aligned} a_{11}\cdot x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0\\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0\\ \vdots \qquad\qquad\qquad\qquad & \ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}}$$

Homogeneous lin. sys. are always consistent

$(A|\underline{0}) \to \dots \to (E|\underline{0})$

and $\underline{x}=\underline{0}$ is called the trivial solution

$(x_1 = x_2 = \dots = x_n = 0)$

Result:

$\mathrm{rk}A = n \iff$ one unique sol'n $\underline{x}=\underline{0}$

$\mathrm{rk}A < n \iff$ inf. many solution (non-trivial solutions)

$\#$ free var $= n - \mathrm{rk}A > 0$

Defn: A $m\times n$ matrix

$\mathrm{Null}(A)$ = the set of all solutions $= \{\underline{x} : A\underline{x}=\underline{0}\}$
nullspace of A of the homogeneous lin. system with aug. matrix $(A\,|\,0)$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 3 & -3 & 6 & 1 \end{pmatrix}$ Null(A) = ?

$$\left.\begin{aligned} x+y+z+w &= 0 \\ 2x-y+3z &= 0 \\ 3x-3y+6z+w &= 0 \end{aligned}\right\} \quad (A|\underline{0}) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 \\ 3 & -3 & 6 & 1 & 0 \end{array}\right) \begin{matrix} \\ -2 \\ -3 \end{matrix} \to$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & -6 & 3 & -2 & 0 \end{array}\right) \begin{matrix} \\ \\ -2 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \quad \operatorname{rk} A = 3 < \underset{n}{4}$$

echelon form

one free var, inf. many soln's

$\Rightarrow$ non-trivial solutions

$$\begin{aligned} x+y+z+w &= 0 \\ -3y+z-2w &= 0 \\ z+2w &= 0 \end{aligned}$$

w free

$z = -2w$

$-3y = 2w - (-2w)$

$\frac{-3y}{-3} = \frac{4w}{-3}$

$y = -\frac{4w}{3}$

$x = -w - (-2w) - (-\frac{4}{3}w)$

$= -w + 2w + \frac{4}{3}w = \frac{7}{3}w$

$(x,y,z,w) = \left(\frac{7}{3}w, -\frac{4}{3}w, -2w, w\right)$

(w free)

$= \left(\frac{7}{3}t, -\frac{4}{3}t, -2t, t\right)$

(t parameter)

non-trivial solution

$w=3: \quad (7, -4, -6, 3)$

line in $\mathbb{R}^4$ thr. the origin

$\text{Null}(A) = \left\{ \left(\frac{7}{3}w, -\frac{4}{3}w, -2w, w\right) \mid w \text{ free / parameter / has any value} \right\}$