Key Problems

Problem 1.
Write the systems of differential equations on matrix form and solve them:

a) $\begin{aligned} y_1' &= 2y_1 + 5y_2 \\ y_2' &= 5y_1 + 2y_2 \end{aligned}$ b) $\begin{aligned} y_1' &= y_2 \\ y_2' &= 4y_1 + 3y_2 \end{aligned}$ c) $\begin{aligned} y_1' &= y_1 + 4y_2 + 3 \\ y_2' &= y_1 - 2y_2 - 3 \end{aligned}$

Problem 2.
Solve the systems of differential equations:

a) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ b) $\mathbf{y}' = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & -1 & 1 \end{pmatrix} \cdot \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} -1 \\ -3 \\ 8 \end{pmatrix}$

Problem 3.
Rewrite the differential equation $y''' + 4y'' + y' - 6y = 0$ as a system of first order linear differential equations, and solve the system of differential equations.

Problem 4.
Let $y(t) = 3e^{-2t} - 5e^{t} + 12e^{-3t}$.

a) Find a linear second order differential equation that has y as a particular solution.
b) Find a linear third order differential equation that has y as a particular solution.
c) Find a 3×3 matrix A such that $\mathbf{y}' = A\mathbf{y}$ has $\mathbf{y} = (y, y', y'')$ as a particular solution.

Problem 5.
Find the equilibrium states and determine their stability:

a) $y'' + 7y' + 10y = 5$ b) $y'' + y' - 20y = 1$ c) $y''' + 4y'' + y' - 6y = 12$

d) $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ e) $\mathbf{y}' = \begin{pmatrix} -5 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \mathbf{y}$

Exercise Problems

Problems from the textbook	[E] 9.1 - 9.7
Final exam problems	11/2018 Q2,Q5, 01/2019 Q2, 01/2020 Q3, 03/2021 Q3bc

Answers to Key Problems

Problem 1.

a) $\mathbf{y}' = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$

b) $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = C_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

c) $\mathbf{y}' = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}, \quad \mathbf{y} = C_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Problem 2.

a) $\mathbf{y} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-4t} - \frac{1}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-6t}$

b) $\mathbf{y} = \begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} e^{3t}$

Problem 3.

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix} \mathbf{y}, \quad \mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} e^{-2t} + C_3 \begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix} e^{-3t}$$

Problem 4.

More than one solution is possible:

a) $y'' + y' - 2y = 48e^{-3t}$

b) $y''' + 4y'' + y' - 6y = 0$

c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -1 & -4 \end{pmatrix}$

Problem 5.

a) $\mathbf{y}_e = 1/2$ is globally asymptotically stable

b) $\mathbf{y}_e = -1/20$ is unstable

c) $\mathbf{y}_e = -2$ is unstable

d) $\mathbf{y}_e = (1, -1)$ is unstable

e) $\mathbf{y}_e = (0,0,0)$ is globally asymptotically stable