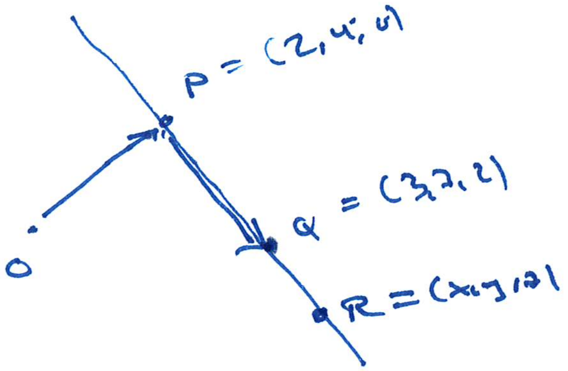
	Plan	
1	Key Problems:	6.1, 6.2a, 6.3c, 6.5b
2	Mock Midterm:	2,4,5,6,7,8,9,10,11,14,17,19,21,22
3	Midterm exams:	Midterm 01/2020 Q7

② Mock midterm



2. $P = (2,4,0)$
$Q = (3,7,2)$

$\vec{OP} + t \cdot \vec{PQ} = (2,4,0) + t \cdot (1,3,2)$

$= (2+t, 4+3t, 2t)$

$x = 1: \quad 2+t = 1 \quad t = -1$
$y = 1: \quad 4+3t = 1 \quad$ —〃—

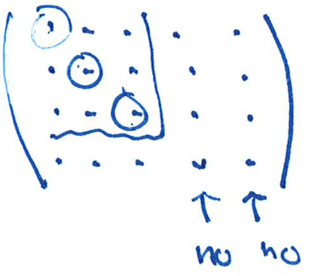
$\left.\right\} \quad \underline{\underline{(1,1,-2)}}$

4. $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 3 \\ 5 & 4 & 6 & 1 \end{array}\right) \begin{matrix} \\ \leftarrow -2 \\ \leftarrow -5 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 1 \\ 0 & \textcircled{-1} & -4 & 1 \\ 0 & -1 & -4 & -4 \end{array}\right) \begin{matrix} \\ \\ \leftarrow -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 1 \\ 0 & \textcircled{-1} & -4 & 1 \\ 0 & 0 & 0 & \textcircled{-5} \end{array}\right)$

no sol's $\Rightarrow$ $\underline{\underline{\text{no}}}$

5. $\text{proj}_{\underline{w}}(\underline{v}) = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \cdot \underline{w} = \frac{4}{4} \cdot \underline{w} = \underline{\underline{(1,1,1,1)}}$

6.



pivots: $(1,1), (2,2), (3,3)$ since $M_{123,123} \neq 0$
no pivot in $(4,4)$ 〃 $M_{1234,1234} = 0$
—〃— $(4,5)$ 〃 $M_{1234,1235} = 0$

$\dim \text{Null}(A) = 5 - 3 = \underline{\underline{2}}$

7.

$$A = \begin{pmatrix} t & 3 \\ 2 & 6 \\ 3 & t \\ 5 & 9+t \end{pmatrix}$$

$M_{12,12} = 6t - 6 = 0 \qquad t = 1$

$M_{23,12} = 2t - 18 = 0 \qquad t = 9$

$\operatorname{rk} A = 2$ for all t

$\Rightarrow \{\underline{v}_1, \underline{v}_2\}$ lin independent for all t

8.

$$-\lambda^3 + \underbrace{\operatorname{tr}(A)}_{14}\lambda^2 + (M_{12} + M_{23} + M_{13})\lambda + \underbrace{|A|}_{0} = 0$$

$$M_{12} + M_{23} + M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 5 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 8 \end{vmatrix} = 7 + 7 + 7 = 21$$

$|A| = 0$ since $R(3) = R(1) + R(2)$

$$\underline{-\lambda^3 + 14\lambda^2 - 21\lambda = 0}$$

9. A regulær

$\underline{E_1}$:

$$\begin{pmatrix} -0.6 & 0.2 & 0.2 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.3 \end{pmatrix} \begin{matrix} \cdot 10 \\ \cdot 10 \\ \cdot 10 \end{matrix} \rightarrow \begin{pmatrix} 2 & 2 & -3 \\ 4 & -4 & 1 \\ -6 & 2 & 2 \end{pmatrix} \begin{matrix} \\ -2 \\ 3 \end{matrix}$$

$$\rightarrow \begin{pmatrix} \enclose{circle}{2} & 2 & -3 \\ 0 & \enclose{circle}{-8} & 7 \\ \cancel{0} & \cancel{8} & \cancel{-7} \end{pmatrix} \qquad \begin{aligned} 2x + 2y - 3z &= 0 \\ -8y + 7z &= 0 \end{aligned}$$

$y = 7z/8$

$2x = 3z - 2(7z/8) = \frac{12z}{4} - \frac{7z}{4}$

$x = \frac{5z}{8}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5z/8 \\ 7z/8 \\ z \end{pmatrix} = \frac{z}{8}\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$$

$$\Rightarrow \underline{v} = \underline{\underline{\frac{1}{20}\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}}} \quad \text{since } 5+7+8 = 20$$

$\underline{10.}$ Eigenvalues $\lambda = 3, s, 1$

$s \neq 1, 3$: diagonalizable

$\underline{s=1}$: $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \underline{\lambda = 1 \quad m = 2:}$ $\begin{pmatrix} \enclose{circle}{2} & 0 & 1 \\ 0 & 0 & \enclose{circle}{2} \\ 0 & 0 & 0 \end{pmatrix}$ one free var. $\Rightarrow \underline{\underline{\text{no}}}$

$\underline{s=3}$: $\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \underline{\lambda = 3 \quad m = 2:}$ $\begin{pmatrix} 0 & 0 & \enclose{circle}{3} \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ two free var. $\Rightarrow \underline{\underline{\text{yes}}}$.

Diag for all $\underline{\underline{s \neq 1}}$

$\underline{11.}$ $\dim E_0 = \dim \text{Null}(A - 0 \cdot I) = \dim \text{Null}(A) = 2$

$\text{rk} A = \underline{\underline{1}} \Leftarrow 3 - \text{rk} A = 2$

$\underline{14.}$ $V = \text{Null}\begin{pmatrix} 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \updownarrow$

$$\downarrow$$

$$\begin{pmatrix} \enclose{circle}{1} & 0 & 1 & 2 & 3 \\ 0 & \enclose{circle}{1} & -1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{1} & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ -1 \quad -1 \\ \\ \end{matrix}$$

$$\downarrow$$

$$\begin{pmatrix} \enclose{circle}{1} & 0 & 1 & 2 & 3 \\ 0 & \enclose{circle}{1} & -1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{1} & 2 & 1 \\ 0 & 0 & 1 & -2 & -2 \end{pmatrix} \begin{matrix} \\ \\ \\ -1 \end{matrix} \rightarrow \begin{pmatrix} \enclose{circle}{1} & 0 & 1 & 2 & 3 \\ 0 & \enclose{circle}{1} & -1 & 1 & 0 \\ 0 & 0 & \enclose{circle}{1} & 2 & 1 \\ 0 & 0 & 0 & -\enclose{circle}{4} & -3 \end{pmatrix}$$

$\dim V = 5 - 4 = \underline{\underline{1}}$

18. $$A-\lambda I = A+I = \begin{pmatrix} \boxed{3} & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

$\text{rk}(A-\lambda I) = \text{rk}(A+I) = \underline{1} \Rightarrow \lambda = -1$ is an $\underline{\text{eigenvalue}}$

$\dim E_{-1} = 4 - \text{rk}(A+I) = 4-1 = \underline{3} \Rightarrow \underline{\underline{m=3}}$

since A is symm.

<u>Alt:</u> $\underset{\overset{\shortparallel}{3}}{\dim E_{-1}} \leq m \Rightarrow m \geq 3$

$\lambda_1 = \lambda_2 = \lambda_3 = -1 \quad \lambda_4 = 11$

$\text{tr}(A) = \lambda_1+\lambda_2+\lambda_3+\lambda_4$

$8 = 3\cdot(-1) + 11$

$\Rightarrow \underline{\underline{m=3}}$ since $\lambda_4 \neq -1$

19. $$\text{Null}\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \text{Null}\begin{pmatrix} \boxed{3} & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x+y+z=0$

y, z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix} = y\cdot\underset{\overset{\shortparallel}{\underline{w}_1}}{\underline{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}} + z\cdot\underset{\overset{\shortparallel}{\underline{w}_2}}{\underline{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}}$$

$\underline{w}_1 \cdot \underline{w}_2 = 1 \neq 0$

$\text{proj}_{\underline{w}_1} \underline{w}_2$

$$\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \underline{w}_1 \qquad \underline{v}_2 = \underline{w}_2 - \frac{\underline{w}_2 \cdot \underline{w}_1}{\underline{w}_1 \cdot \underline{w}_1} \cdot \underline{w}_1$$

$$= (-1,0,1) - \frac{1}{2}(-1,1,0) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}, 1\right)$$

$$= \frac{1}{2}\underline{(-1,-1,2)}$$

$\Rightarrow$ <u>Orth base:</u> $$\underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}}}$$

21. $f = x^4 + y^4 + z^4 - 4xyz$

$$\left.\begin{aligned} f'_x &= 4x^3 - 4yz = 0 \\ f'_y &= 4y^3 - 4xz = 0 \\ f'_z &= 4z^3 - 4xy = 0 \end{aligned}\right\} \quad (1,1,1) \text{ is a solution}$$

$$\underset{(1,1,1)}{H(f)} = \begin{pmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{pmatrix} (1,1,1) = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{pmatrix}$$

pos defn

$\Downarrow$

(1,1,1)

local min

$D_1 = 12$

$D_2 = 144 - 16 = 128$

$D_3 = -4\cdot(16+48) + 4\cdot(-48-16) + 12\cdot128$

$= -4\cdot64 - 4\cdot64 + 12\cdot128$

$= -8\cdot64 + 24\cdot64 > 0$

22. $f = 3 - a\cdot g(x,y,z,w)$

$H(f) = 0 - a\cdot H(g) = -a\cdot H(g)$

$a > 0$: $H(g)$ neg. semidefn. $-a < 0$ } $H(f)$ pos. semidefn. f convex

$a < 0$: $H(g)$ —"— $-a > 0$ } $H(f)$ neg. semidefn. f concave

$a = 0$: $H(f) = 0$ f convex and concave

③ Midterm 01/2020, Q7

$$f = x^4 + y^4 + z^4 - 4xy$$

$$f'_x = 4x^3 - 4y = 0 \Rightarrow x^3 - y = 0 \Rightarrow \boxed{y = x^3}$$

$$f'_y = 4y^3 - 4x = 0 \qquad y^3 - x = 0 : (x^3)^3 - x = 0$$

$$f'_z = 4z^3 = 0 \qquad \underline{z = 0}$$

$$x^9 - x = 0$$

$$x \cdot (x^8 - 1) = 0$$

$$\underline{x = 0} \text{ or } x = \pm \sqrt[8]{1} = \underline{\pm 1}$$

Classify:

$\Rightarrow$ Stationary pts: $(0,0,0)$, $(1,1,0)$, $(-1,-1,0)$

$$H(f) = \begin{pmatrix} 12x^2 & -4 & 0 \\ -4 & 12y^2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

$\underline{(0,0,0)}$: $H(f)(0,0,0) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $D_1 = 0$, $D_2 = -16$ indefin. $\Rightarrow$ $(0,0,0)$ saddle pt

$\underline{(\pm1,\pm1,0)}$: $H(f)(\pm1,\pm1,0) = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $D_1 = 12$, $D_2 = 144 - 16 > 0$, $D_3 = 0$

RRC (rk = 2) $\Downarrow$ pos. semidefin.

$$f(x,y,z) = f_1(x,y) + f_2(z) = \underbrace{x^4 + y^4 - 4xy}_{f_1} + \underbrace{z^4}_{f_2}$$

Second derivative test is inconclusive

$$f'_{1x} = 4x^3 - 4y = 0$$

$$f'_{1y} = 4y^3 - 4x = 0$$

Stat. pts: $\underline{(0,0)}$, $(1,1)$, $(-1,-1)$

$(1,1)$, $(-1,-1)$ loc. min for f_1

$$H(f_1) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix} \qquad H(f_1)(\pm1,\pm1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}$$ pos. defin.

$D_1 = 12$ $D_2 = 144 - 16$

$f(z) = z^4$

$z = 0$ loc. min of f_2

$\Downarrow$

$(\pm1,\pm1,0)$ is local min for f.

① Key problem Lecture 6

1. $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & a & -a & 0 \\ 0 & -a & a & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

$D_1 = 1$
$D_2 = a$
$D_3 = 1(a^2 - a^2) = 0$
$D_4 = 0$

RRC:

$\text{rk } A = \begin{cases} 2 & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \end{cases}$

$a > 0$: $D_1, D_2 > 0 \overset{\text{rk}=2}{\underset{}{\overset{\text{RRC}}{\Rightarrow}}}$ pos. semidefin.

$a < 0$: $D_2 < 0 \Rightarrow$ indefinite

$a = 0$: $D_1 > 0 \underset{\substack{\text{RRC} \\ \text{rk}=1}}{\Rightarrow}$ pos. semidefin.

2. a) $f(\underline{x}) = \underline{x}^T A \underline{x}$

$A = \begin{array}{c|ccc|c} & x & y & z & \\ \hline & \boxed{5} & 3 & 8 & x \\ & 3 & 2 & 5 & y \\ & 8 & 5 & 13 & z \end{array}$

$D_1 = 5$
$D_2 = 1$
$D_3 = 0$

$f'(x) = 2A \cdot \underline{x} = \underline{0}$ ← one free var, $\underline{x} = \underline{0}$ is one of the stationary pts.

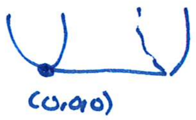
$f''(x) = 2A$

A pos. semidefin. by RRC

Since $H(x)$ is pos. semidefin. at all pts $\underline{x}$,

f is convex $\Rightarrow$ (0,0,0) global min

$f_{\min} = f(0,0,0) = \underline{0}$



3. c) $f = x^4 + y^4 + z^4 + z^2$

$$\left.\begin{array}{ll} f'_x = 4x^3 = 0 & \underline{x=0} \\ f'_y = 4y^3 = 0 & \underline{y=0} \\ f'_z = 4z^3 + 2z = 0 & \end{array}\right\}$$

Stat. pts: $\underline{(0,0,0)}$

$2z(2z^2+1) = 0 \quad \underline{z=0} \text{ or } \cancel{2z^2+1=0}$

$$H(f) = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2+2 \end{pmatrix}$$

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

pos. semidefn. <u>inconclusive test.</u>

<u>Convex?</u> $H(f) = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2+2 \end{pmatrix}$

$\lambda_1 = 12x^2 \geq 0$

$\lambda_2 = 12y^2 \geq 0$

$\lambda_3 = 12z^2+2 > 0$

pos. semidefn at <u>all pts</u>

$\Rightarrow f$ <u>convex</u> $\Rightarrow (0,0,0)$ <u>global min</u>

$f_{min} = f(0,0,0) = \underline{\underline{0}}$

5. a) $f(x,y,z) = \ln(1+\underline{2x^2+2xy+3y^2-2xz+z^2})$

$= \ln(1+u),\ u = 2x^2+2xy+3y^2-2xz+z^2$

u = inner fn.

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \begin{matrix} D_1 = 2 \\ D_2 = 5 \\ D_3 = 2 \\ \| \\ -1\cdot 3 + 1\cdot 5 \end{matrix}$$

pos. defn.

$u'(\underline{x}) = 2A\cdot\underline{x} = \underline{0} \Rightarrow \underline{\underline{x} = \underline{0}}$

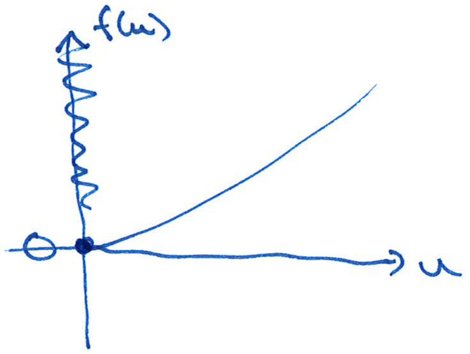
u convex since $H(u) = 2A$ is pos. defn.

$u_{min} = u(0,0,0) = 0$

$V_u = [0, \rightarrow) \Rightarrow u \geq 0$

outer fn $f(u) = \ln(1+u),\ u \geq 0$

$f'(u) = \frac{1}{1+u}\cdot 1 = \frac{1}{1+u} > 0$



$f_{min} = 0 \qquad \lim_{u\to\infty} \ln(1+u) = \infty$

$$\underline{\underline{V_f = [0, \rightarrow)}}$$

b) $f = (x^2+y^2+z^2)\cdot e^{-x^2-y^2-z^2}$

$= u\cdot e^{-u},\ u = x^2+y^2+z^2$

$u = x^2+y^2+z^2$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{pos. defn.}$$

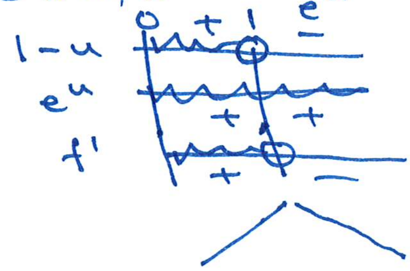
$u_{min} = u(0,0,0) = 0$

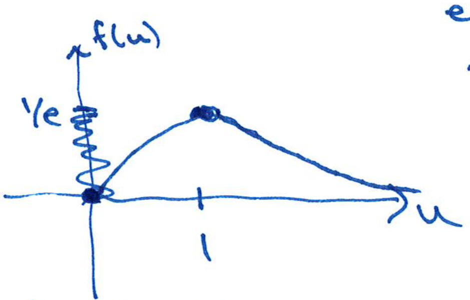
$V_u = [0, \rightarrow) \Leftrightarrow u \geq 0$

$f(u) = ue^{-u},\ u \geq 0$

$f'(u) = 1\cdot e^{-u} + u\cdot e^{-u}\cdot(-1)$

$= e^{-u} - ue^{-u} = (1-u)e^{-u} = \underline{\frac{1-u}{e^u}}$





$\lim_{u\to\infty} ue^{-u} = \lim_{u\to\infty} \frac{u}{e^u} = 0$

$$\underline{\underline{V_f = [0, 1/e]}}$$