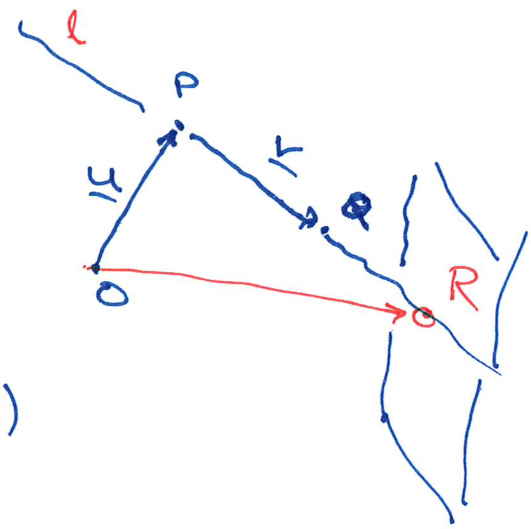
	Plan	
1	Key Problems:	2.2, 2.3a, 2.4ef, 3.1, 3.3, 3.4, 4.3, 4.4, 4.5
2	Textbook Problems:	4.8b, 2.2b
3	Midterm Exams:	01/2022 4,5 , 10/21 Q1-5, 8

① Key Problems



2.2 $P = (1,3,2,5)$
$Q = (-2,4,5,1)$

$$\vec{OP} = \underline{u} = (1,3,2,5)$$
$$\vec{PQ} = \underline{v} = (-2-1, 4-3, 5-2, 1-5) = (-3,1,3,-4)$$

R in l: (x,y,z,w)

$$\vec{OR} = \vec{OP} + \vec{PR} = \underline{u} + t\cdot\vec{PQ} = \underline{u} + t\cdot\underline{v}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix} + t\cdot\begin{pmatrix} -3 \\ 1 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1-3t \\ 3+t \\ 2+3t \\ 5-4t \end{pmatrix}$$

$(x,y,z,w) = (1,3,2,5) + t(-3,1,3,-4)$

$x+z+w=0$: $(1-3t)+(2+3t)+(5-4t)=0$

$8-4t=0 \quad t=2 \Rightarrow (x,y,z,w) = (-5,5,8,-3)$

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2.3 a)

$$A=\begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 2 & 4 & -2 & -2 & -2 \\ 3 & 5 & -1 & -1 & -1 \end{pmatrix} \begin{matrix} -2 \\ -3 \end{matrix} \qquad \mathrm{Null}(A) = ?$$

$-14\cdot(-4/3) = 56/3$

$-10\cdot(-4/3) = 40/3$

$$\to \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 8 & -16 & -19 & -13 \end{pmatrix} -4/3 \to \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 0 & 0 & -1/3 & 1/3 \end{pmatrix} \cdot 3$$

$$\to \begin{pmatrix} 1 & -1 & 5 & 6 & 4 \\ 0 & 6 & -12 & -14 & -10 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$\dim \mathrm{Null}(A) = 5 - \mathrm{rk}\,A = 5-3 = 2$

Base: x_3, x_5 free

(3) $-x_4 + x_5 = 0 \Rightarrow x_4 = x_5$

(2) $6x_2 - 12x_3 - 14x_5 - 10x_5 = 0$

$$\frac{6x_2}{6} = \frac{12x_3 + 24x_5}{6}$$

$x_2 = 2x_3 + 4x_5$

(1) $x_1 - (2x_3 + 4x_5) + 5x_3 + 6x_5 + 4x_5 = 0$

$x_1 = -3x_3 - 6x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -3x_3 - 6x_5 \\ 2x_3 + 4x_5 \\ x_3 \\ x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \underbrace{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{= w_1} + x_5 \cdot \underbrace{\begin{pmatrix} -6 \\ 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{= w_2}$$

Base of Null(A): $\{w_1, w_2\}$

<u>24</u> e) Solution of $A\underline{x}=\underline{b}$:

A 8×8, $\underline{b}$ 8-vector, $rk(A)=7$

$(A|\underline{b}) \to \dots \to (E|\underline{c})$ echelon form

last row $(0\,0\dots 0\,|\,*)$

$rk(A|\underline{b})=7$: one free variable, inf. many solutions

$rk(A|\underline{b})=8$ no solutions

f) Solutions of $A\underline{x}=\underline{0}$ that satisfies $x_1+x_2+\dots+x_8=1$

$$\left(\begin{array}{c|c} A & \underline{0} \\ \hline 1\,1\dots 1 & 1 \end{array}\right) \to \left(\begin{array}{c|c} E & \underline{0} \\ \hline 0\,0\dots 0\,* & 1 \end{array}\right)$$

$* = 0$: no solutions

$* \neq 0$: one solution

<u>3.1</u>

a) $A=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ $|A| = 1\cdot(1-4)-2\cdot(0-2) = -3+4=\underline{1}$

$$A^{-1}=\frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{1}\begin{pmatrix} -3 & 2 & -1 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix}^T = \underline{\underline{\begin{pmatrix} -3 & -2 & 4 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix}}}$$

$$A^2=\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 6 & 5 \end{pmatrix}}}$$

3.3 b) $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$

$M_{123,123} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 7 \\ 4 & 5 & 11 \end{vmatrix} = 1(-46) - 3(22-28) + 2(10+4)$

$= -46 + 18 + 28 = 0$

$M_{123,124} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 4 & 5 & 10 \end{vmatrix} = 1 \cdot (-10-15) - 3(20-12) + 4 \cdot (10+4)$

$= -25 - 24 + 56 = 7 \neq 0$

$\text{rk}\, A = 3$ Base of Col(A): Col. 1, 2, 4

Base of Row(A): Rows 1, 2, 3

3.4 c) $A = \begin{pmatrix} 1 & a & b \\ a & b & c \end{pmatrix}$ $\boxed{\text{rk}\, A < 2 \iff \text{all 2-minors} = 0}$

$b = a^2$: $\Leftarrow$ $b - a^2 = 0$ $M_{12,12} = \begin{vmatrix} 1 & a \\ a & b \end{vmatrix} = 0$

$ac - a^4 = 0$ $ac - b^2 = 0$ $M_{12,23} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} = 0$

$c - a^3 = 0$ $c - ab = 0$ $M_{12,13} = \begin{vmatrix} 1 & b \\ a & c \end{vmatrix} = 0$

$c = a^3$

$\Downarrow$

$\text{rk}\, A = 1$ or ~~$\text{rk}\, A = 0$~~ $\leftarrow$ $\text{rk}\, A < 2 \iff b = a^2$ and $c = a^3$

$\Downarrow$

$$\text{rk}\, A = \begin{cases} 1 & , b = a^2 \text{ and } c = a^3 \\ 2 & , \text{otherwise} \end{cases}$$

4.3

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

i) Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 0 & 0 & 4 \\ 0 & 2-\lambda & 3 & 0 \\ 0 & 3 & 2-\lambda & 0 \\ 4 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

ii) Diagonalisable

A symmetric $\Rightarrow$ A diagonalizable

Alt: i) $n=4$ eigenvalues c/multiplicity ✓

ii) $n=4$ lin. independent eigenvectors

check that $\dim E_5 = 2$ ✓

$$\underbrace{(1-\lambda)\cdot \begin{vmatrix} \boxed{\begin{matrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1-\lambda \end{vmatrix}} - \underbrace{4 \cdot \begin{vmatrix} \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{matrix}} \\ 4 & \begin{matrix} 0 & 0 \end{matrix} \end{vmatrix}} = 0$$

$$(1-\lambda)\ (1-\lambda)\cdot \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} - 4\cdot 4 \cdot \begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda^2 - 4\lambda - 5)\cdot\left[(1-\lambda)^2 - 16\right] = 0$$

$$(\lambda^2 - 4\lambda - 5)\cdot(\lambda^2 - 2\lambda - 15) = 0$$

$\lambda^2 - 4\lambda - 5 = 0$ or $\lambda^2 - 2\lambda - 15 = 0$

$(\lambda - 5)(\lambda + 1) = 0$ $\qquad (\lambda - 5)(\lambda + 3) = 0$

$\lambda_1 = 5,\ \lambda_2 = -1$ $\qquad \lambda_3 = 5,\ \lambda_4 = -3$

$$\begin{cases} \lambda = 5 \ (\text{mult } 2) \\ \lambda = -1 \\ \lambda = -3 \end{cases}$$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \operatorname{tr}(A)$

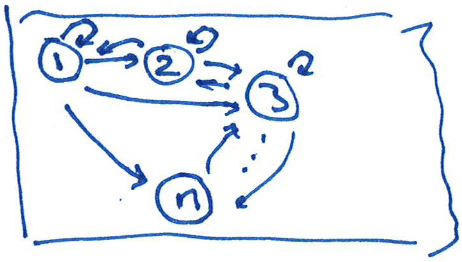
$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 = \det(A)$

$\dim E_{-1} = 1$

$\dim E_{-3} = 1$

$1 \le \dim E_5 \le 2$

Markov chains:



n states

Defn: $\underline{x} = (x_1, x_2, \ldots, x_n)$ is a <u>state vector</u> if

i) $x_i \geq 0$ for all i

ii) $x_1 + x_2 + \ldots + x_n = 1$

$A = (a_{ij})$ is an $n \times n$ <u>transition matrix</u> if

i) $a_{ij} \geq 0$ for all i,j

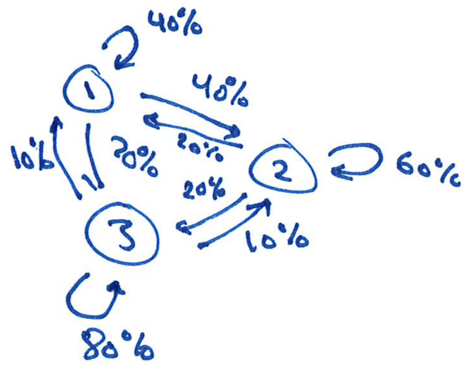
ii) the sum in each column should be 1

Markov chain:

$$\underline{x}_{t+1} = A \cdot \underline{x}_t$$

Defn: The Markov chain is <u>regular</u> if you can get from any state to any other state in m steps for some $m \geq 1$.

Ex:
$$A = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{pmatrix}$$



Thm: Let $\underline{x}_{t+1} = A\underline{x}_t$ be a <u>regular Markov chain</u> with transition matrix A.

Then i) $\lambda = 1$ is one eigenvalue of A, and there is a unique eigenvector $\underline{v}$ in E_1 that is a state vector

ii) $\lim_{N \to \infty} A^N = (\underline{v} | \underline{v} | \ldots | \underline{v})$ and $\lim_{N \to \infty} A^N \cdot \underline{v}_0 = \underline{v}$

for any initial state vector $\underline{v}_0$.

$\underline{v}$ is called the equilibrium state of the Markov chain

4.4 a) $A = \begin{pmatrix} 0.40 & 0.15 \\ 0.60 & 0.85 \end{pmatrix}$ regular Markov chain

Theory:

$\lambda = 1$: $E_1 = \text{Null}\begin{pmatrix} -0.60 & 0.15 \\ \cancel{0.60} & \cancel{-0.15} \end{pmatrix}$ $\qquad -0.60x + 0.15y = 0$

y free

$x = \dfrac{-0.15y}{-0.60} = y/4$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/4 \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$

$\underline{V} = \frac{4}{5} \cdot \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$

eq. state

State vector: $y/4 + y = 1$

$y(1/4 + 1) = 1$

$y \cdot 5/4 = 1 \qquad y = 4/5$

$\lim_{m \to \infty} A^m = \begin{pmatrix} 1/5 & 1/5 \\ 4/5 & 4/5 \end{pmatrix} \qquad \lim_{m \to \infty} A^m \cdot \underline{V_0} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$

$\left.\begin{array}{l} \underline{V_0} = \begin{pmatrix} x \\ y \end{pmatrix} \\ x + y = 1 \end{array}\right\} \quad \begin{pmatrix} 1/5 & 1/5 \\ 4/5 & 4/5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x \cdot \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix} + y \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$

$= \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}$

Eigenvalues: $\lambda = 1$, $\lambda = 0.25$

$D^m = \begin{pmatrix} 1 & 0 \\ 0 & 0.25 \end{pmatrix}^m = \begin{pmatrix} 1^m & 0 \\ 0 & 0.25^m \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 0.77 & 0.46 \\ 0.23 & 0.54 \end{pmatrix}$

regular

$\begin{pmatrix} -0.23 & 0.46 \\ \cancel{0.23} & \cancel{-0.46} \end{pmatrix}$ $-0.23x + 0.46y = 0$

y free

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $x = 2y$

y free

$\Rightarrow \underline{v} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ eq. state

2 Textbook problems

4.8 b)

$A = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{pmatrix}$ $E_1 = \text{Null}\begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix}$

regular

$\rightarrow \begin{pmatrix} 2 & 2 & -2 \\ 4 & -4 & 1 \\ -6 & 2 & 1 \end{pmatrix} \begin{matrix} -2 \\ 3 \end{matrix} \rightarrow \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ \cancel{0} & \cancel{8} & \cancel{-5} \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/8 \\ 5/8 \\ 1 \end{pmatrix} \cdot z$

(2) $-8y + 5z = 0$

$y = \frac{5}{8} z$

(1) $x + (5/8\, z) - z = 0$

$x = z - 5/8\, z = 3/8\, z$

$\rightarrow \underline{v} = \frac{1}{2}\begin{pmatrix} 3/8 \\ 5/8 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/16 \\ 5/16 \\ 8/16 \end{pmatrix}$

eq. state

③ Midterm exam 01/2022, Q5

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & s & 2 \\ 1 & 0 & 2 \end{pmatrix}$$ When is A diagonalizable?

Eigenvalues: $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & s-\lambda & 2 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$+ (s-\lambda)\cdot\left((2-\lambda)^2 - 1^2\right) = 0$$

$$(s-\lambda)\cdot(\lambda^2 - 4\lambda + 3) = 0$$

$\lambda = s$ or $\lambda^2 - 4\lambda + 3 = 0$

$\lambda = 3,\ \lambda = 1$

n=3 eigenvalues c/ with mult.

$s=1$: $E_1 = \text{Null}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ \cancel{1} & \cancel{0} & \cancel{1} \end{pmatrix}$ y free, $\dim E_1 = 1 < m = 2$ No

$\lambda = 1, 1, 3$ — $m = 2$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $P = \begin{pmatrix} | & | & | \end{pmatrix}$ $\lambda=1$ $\lambda=3$

$s=3$: $E_3 = \text{Null}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 2 \\ \cancel{1} & \cancel{0} & \cancel{-1} \end{pmatrix}$ y free, $\dim E_3 = 1 < m = 2$ No

$\lambda = 3, 3, 1$

$s \neq 1, 3$: n distinct eigenvalues $\Rightarrow$ A diagonalizable Yes

Key Problem 4.5

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$|A - \lambda I| = \underbrace{-\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3 = 0}_{\text{char eqn. of a } 3\times 3 \text{ matrix}}$$

$c_1 = tr(A) = a + e + i$

Sum of all principal 2-minors $\rightarrow$ $c_2 = M_{12,12} + M_{23,23} + M_{13,13}$

$c_3 = det(A)$

a) $$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 0 \\ 3 & 5 & 1 \end{pmatrix}$$

$-\lambda^3 + 6\lambda^2 - 4\lambda + 0 = 0$

$-\lambda(\lambda^2 - 6\lambda + 4) = 0$

$\lambda = 0$ or $\lambda = \dfrac{6 \pm \sqrt{36 - 16}}{2}$

$= 3 \pm \dfrac{\sqrt{20}}{2}$

$M_{12,12} = 2$

$M_{23,23} = 4$

$M_{13,13} = -2$

$|A| = 1 \cdot (-2) + 1 \cdot 2$

$= 0$

$\underline{\lambda = 0}$, $\underline{\lambda = 3 \pm \sqrt{5}}$