Plan

1 Unconstrained optimization
2 Convex and concave functions
3 Envelope theorems

Plenary Session 2: 10/10
Midterm Exam: 14/10

1 Unconstrained optimization

max/min $f(x_1, x_2, \dots, x_n) = f(\underline{x})$

Defn: The domain of definition $D = D_f$ in $\mathbb{R}^n$ is open if $D = \mathbb{R}^n$ or a subset given by open inequalities (< or >).
The function f is C^2 if all its second order derivatives exist and are continuous in D

Assume: f is C^2 on an open set D in $\mathbb{R}^n$

Ex:
$f(x,y,z) = x^3 - xyz$ $\quad D = \mathbb{R}^3$
$f(x,y,z) = e^{x-y+z}$ $\quad$ – " –
$f(x,y,z) = \ln(x-y+z)$ $\quad D: x-y+z>0$

Defn: $\underline{x}^*$ is max. for f if $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ in D
" $\quad$ min for f " $f(\underline{x}^*) \leq f(\underline{x})$ – " –

max = global max
min = global min

Defn: A stationary pt for f is a pt. where $\underbrace{f'_{x_1} = f'_{x_2} = \dots = f'_{x_n} = 0}_{\text{FOC}}$

Result: If $\underline{x}^*$ is a max/min for f, then it is a stationary pt.

Defn: The Hessian matrix is symmetric:

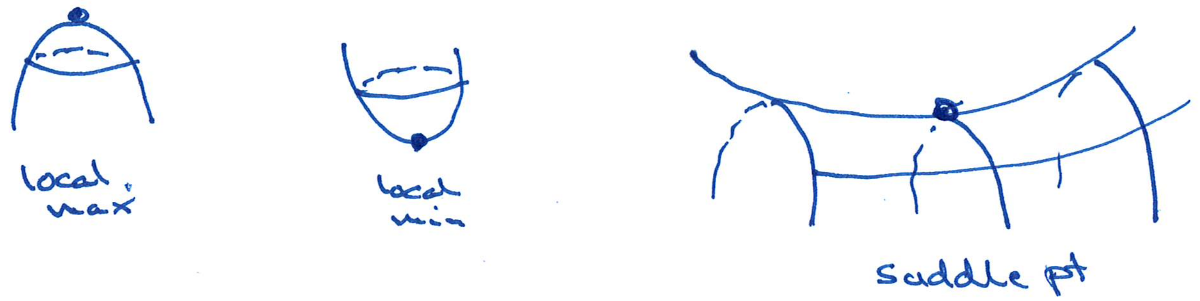
$$H(f) = \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \cdots \\ f''_{x_2x_1} & f''_{x_2x_2} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

Second derivative test: If $\underline{x}^*$ is a stat. pt. for f:
$H(f)(\underline{x}^*)$ pos. defn. $\Rightarrow$ $\underline{x}^*$ local min
" neg. defn. $\quad$ local max
indefinite $\quad$ saddle pt.

Defn: $\underline{x}^*$ is a local max if $f(\underline{x}^*) \geq f(\underline{x})$ for $\underline{x}$ close to $\underline{x}^*$

— " — local min " $f(\underline{x}^*) \leq f(\underline{x})$ — " —

$\underline{x}^*$ is a saddle pt if it is a stationary pt that is neither local max nor local min.



Ex: max/min $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$
$-x-y+z+w-1$

Stationary pts:

FOC:
$f'_x = 2x+y-1=0$
$f'_y = 2y+x+z-1=0$
$f'_z = 2z+y+w+1=0$
$f'_w = 2w+z+1=0$

$$H(f) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

f polynomial of deg 2
⇓
H(f) constant matrix

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 2 & 1 & 0 & 0 & 1 \end{array}\right) -2$$

$$\to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & -3 & -2 & 0 & -1 \end{array}\right) 3 \to \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ \cancel{0} & \cancel{0} & \cancel{4} & \cancel{3} & \cancel{-4} \end{array}\right) -4$$
$0 \; 0 \; 0 \; -5 \quad 0$

$D_1 = \underline{2}$
$D_2 = \underline{3}$
$D_3 = -1 \cdot 2 + 2 \cdot 3 = \underline{4}$
$D_4 = -1 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix} + 2 \cdot D_3 = -1 \cdot 1 \cdot 3 + 2 \cdot 4 = \underline{5}$

H(f) pos. defn.
⇓
$H(f)(\underline{x}^*) = H(f)$ pos. defn.
⇓
any stationary pt. is a local min.

$w=0$
$z=-1$
$y=1$
$x=0$

Stationary pts: $\underline{x}^* = (x,y,z,w) = (0,1,-1,0) \Rightarrow$ local min $\Rightarrow$ is it min? Yes!

f is convex

Quadratic functions:

Polynomial of degree two

For any quadr. function $f(\underline{x})$, we can write

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$$

Ex: $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$
$-x-y+z+w-1$

$$f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & -1 & 1 & 1 \end{pmatrix} \qquad C = -1$$

$n \times n$ symmetric matrix $\qquad$ B $1 \times n$ matrix

Facts:

$$\boxed{\begin{aligned} f'(\underline{x}) &= 2A\underline{x} + B^T \\ f''(\underline{x}) &= 2A \end{aligned}}$$

$$\begin{pmatrix} f'_{x_1} \\ f'_{x_2} \\ \vdots \\ f'_{x_n} \end{pmatrix}$$

$$f''(x) = H(f) = 2A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Result: $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$

i) A positive semidefinite $\Rightarrow$ any stationary pt is global min

ii) " negative — " — — " — global max

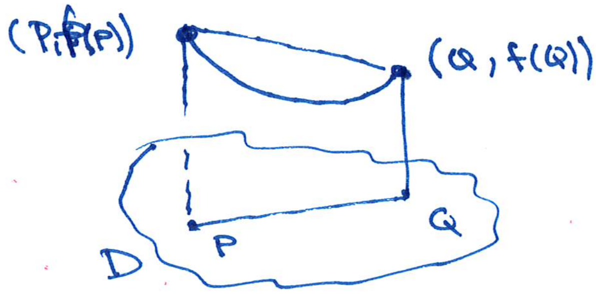
iii) " indefinite — " — saddle pt

② Convex and concave functions

Defn:

$f(\underline{x})$ defined on D,
where D is a convex set

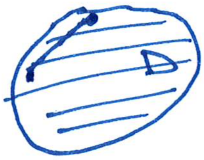
f is convex if the following condition holds:
For any pts $P, Q \in D$, the graph of f along the line segment $[P, Q]$ lies on or under the straight line from $(P, f(P))$ to $(Q, f(Q))$.



Defn:
A subset D of $\mathbb{R}^n$ is called convex if for any pts P, Q in D, the line segment $[P, Q]$ from P to Q lies inside D

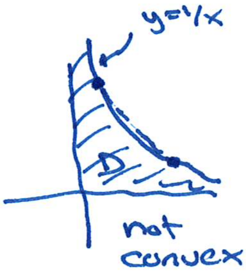
f is concave if $-f$ is convex

Ex:



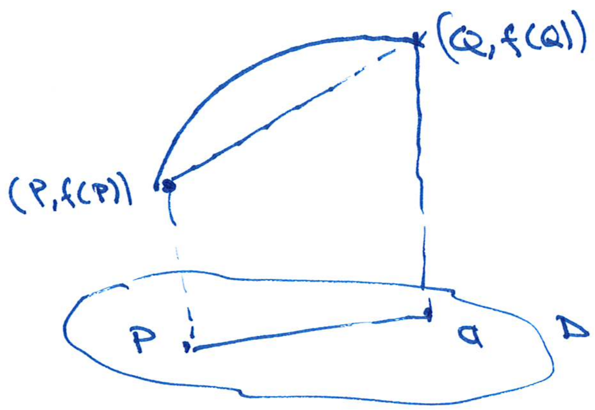
convex

not convex



not convex set

$x \geq 0, y \geq 0$
$xy \leq 1$



Results: f is a C^2 function on an open convex set

f is convex $\Leftrightarrow$ $H(f)(\underline{x})$ is positive semidefn. for all $\underline{x}$ in D

f is concave $\Leftrightarrow$ $H(f)(\underline{x})$ is negative semidefn. — " —

Ex: $f(x,y,z) = x^2y^3 + y^2 - 2y + z^4$, $D = \mathbb{R}^3$

(1) $f'_x = 2xy^3 = 0$

(2) $f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$

(3) $f'_z = 4z^3 = 0$

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 & 0 \\ 6xy^2 & 6x^2y+2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix}$$

(3): $\underline{z=0}$ (1) $2x \cdot y^3 = 0$

$\underline{x=0}$ or $\underline{y=0}$

(2) $2y - 2 = 0$ | $-2 = 0$ impossible

$2y = 2$

$\underline{y=1}$

$\Downarrow$

$(x,y,z) = \underline{(0,1,0)}$

Stat. pts: $\underline{\underline{(0,1,0)}}$

Classification of $(0,1,0)$:

$$H(f)(0,1,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left.\begin{array}{l} D_1 = 2 \\ D_2 = 4 \\ D_3 = 0 \end{array}\right\} \overset{\Rightarrow}{\text{SEC}}$ pos. semidefn. (but not pos. defn.)

test is inconclusive

Is f convex / concave ?

$$H(f) = \begin{pmatrix} \boxed{2y^3} & 6xy^2 & 0 \\ 6xy^2 & 6x^2y+2 & 0 \\ 0 & 0 & 12z^2 \end{pmatrix} \qquad D_1 = 2y^3 \begin{cases} \geq 0 & \text{if } y \geq 0 \\ \leq 0 & \text{if } y \leq 0 \end{cases}$$

f convex $\Leftrightarrow$ $H(f)(\underline{x})$ is pos. semidefin. for all $\underline{x}$ **No.**

f concave $\Leftrightarrow$ — " — neg. — " — **No.**

f is neither convex nor concave

Theorem:
If f is convex, then any stationary pt is a global min.
If f is concave, — " — is a global max.

Note: If $f(\underline{x}) = \underline{x}^T A \underline{x} + B\underline{x} + C$ is quadratic, then the Hessian of f is $H(f) = \underline{2A}$. Therefore:

A positive semidefin $\Rightarrow$ f convex $\Rightarrow$ stat. pts are min

A negative — " — $\Rightarrow$ f concave $\Rightarrow$ stat pts are max

A indefinite $\Rightarrow$ f neither convex nor concave

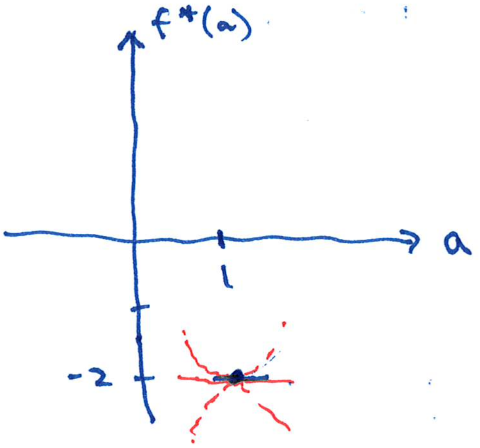
③ Envelope thm

$$\min f(x,y,z,w;a) = x^2+y^2+z^2+w^2 + axy + yz + zw - x - y + z + w - 1$$

$a=1$: Global min. pt: $(0,1,-1,0) = \underline{x}^*(a)$ for $a=1$

Global min val: $f(0,+1,-1,0) = 1-2-1 = \underline{-2} = f^*(a)$ for $a=1$

optimal value fn.



$$\boxed{\textbf{Envelope thm:}\quad \frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))}$$

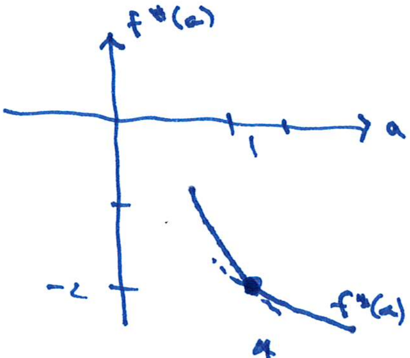
$f'_a = xy$

$f'_a(\underline{x}^*(a)) = f'_a(0,1,-1,0) = 0\cdot 1 = \underline{0}$ at $\underline{a=1}$

$$\min f(x,y,z,w;a) = x^2+y^2+z^2+w^2 + xy + ayz + zw - x - y + z + w - 1$$

$a=1$: $\underline{x}^*(1) = (0,1,-1,0)$

$f^*(1) = f(0,1,-1,0) = \underline{-2}$



$$\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$$

$f'_a = yz \Rightarrow f'_a(x^*(a)) = 1\cdot(-1) = \underline{-1}$ at $a=1$

slope of the tangent line at $a=1$ is $\underline{-1}$

$f^*(1.2) \cong -2 + 0.2\cdot(-1) = -2.2$

Range of a function:

f defined on D in $\mathbb{R}^n$: $V_f = \{f(\underline{x}) : \underline{x} \text{ in } D\}$ the set of possible function values of f

Ex: $f(x,y,z,w) = x^2+y^2+z^2+w^2+xy+yz+zw$
$-x-y+z+w-1$

$f_{min} = \underline{-2}$ no max value ($f \to \infty$) $\Rightarrow V_f = [-2, \to)$