Midterm exam: Oct 14 09-10
pen & paper, written (not multiple ch.)

Plan
1 Orthogonal diagonalization
2 Definiteness of quadratic forms

Review:

$\underset{n\times n}{A}$ λ eigenvalue of A if $A\underline{u}=\lambda\underline{u}$ has nontrivial solutions $\Leftrightarrow$ $|A-\lambda I|=0$ char. eqn.

$\lambda=\lambda_i$ has mult. m $\Leftrightarrow$ $(\lambda-\lambda_i)^m$ factor in $|A-\lambda I|$

λ eigenvalue of A: (mult. m) $E_\lambda = \text{Null}(A-\lambda I)$ eigenspace, set of all eigenvectors of A with eigenvalue λ

$1 \le \dim E_\lambda \le m$

A is diagonalizable if $P^{-1}AP=D$ is diagonal $\Leftrightarrow$
i) there are n eigenvalues of A, counted with multiplicity
ii) there are n linearly independent eigenvectors of A

ii) holds $\Leftrightarrow$ $\dim E_\lambda = m$ for all eigenvalues λ of mult. m

A symmetric $\Rightarrow$ A diagonalizable

$\underset{n\times n}{A}$ $A\cdot\underline{u}_t = \underline{u}_{t+1}$ regular Markov chain

Thm: There is unique vector $\underline{v}$ in E_1 that is a state vector, called the equilibrium state of the Markov chain, and we have

$$\lim_{N\to\infty} A^N = (\underline{v}|\underline{v}|..|\underline{v}), \quad \lim_{N\to\infty} A^N\cdot\underline{u}_0 = \underline{v}$$

A regular $\Leftrightarrow$ $A^m>0$ for some $m\ge 1$ $\Leftrightarrow$ There is an $m\ge1$ such that you can get from any state to any other state in exactly m steps

① Orthogonal diagonalization

$P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$ $n \times n$ matrix

Fact: $P^{-1} = P^T \iff \begin{cases} \text{i) } \underline{v_i} \cdot v_i = 1 & \text{for all } i \\ \text{ii) } \underline{v_i} \cdot v_j = 0 & \text{when } i \neq j \end{cases}$

$I = P^T \cdot P = \begin{pmatrix} \underline{v_1} \\ \underline{v_2} \\ \vdots \\ \underline{v_n} \end{pmatrix} \cdot (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) = \begin{pmatrix} v_1 \cdot v_1 & \dots \\ v_2 \cdot v_1 & \\ \vdots & \end{pmatrix}$

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ orthonormal set

ii) $\underline{v_i} \perp \underline{v_j}$ when $i \neq j$ orthogonal

i) $\| v_i \| = 1$

Defn: A $n \times n$ matrix. An orthogonal diagonalization of A is a diagonalization $P^{-1}AP = D$ such that $P^{-1} = P^T$. $(P^T A P = D)$

Thm: A has an orthogonal diagonalization $\iff$ A is symmetric.

$P^T A P = D \Rightarrow A = PDP^T \Rightarrow A^T = (PDP^T)^T = (P^T)^T \cdot D^T P^T = PDP^T = A$

Ex 1: (easy) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

i) Eigenvalues: $\lambda^2 - 4\lambda + 3 = 0$, $\underline{\lambda = 1}, \underline{\lambda = 3}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Diag: $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

ii) Eigenvectors:

$E_1 = \text{Null}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad x + y = 0$, y free $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$E_3 = \text{Null}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad -x + y = 0$, y free $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(-1, 1) \cdot (-1, 1) = (-1)^2 + 1^2 = 2$ $\quad \| (-1,1) \| = \sqrt{2}$

$(1, 1) \cdot (1, 1) = 1^2 + 1^2 = 2$ $\quad \| (1,1) \| = \sqrt{2}$

$\Downarrow$

$\underline{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad \underline{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{v_1} \cdot \underline{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \cdot (-1 \cdot 1 + 1 \cdot 1) = 0$

Orthogonal diagonalization:

$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$P = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$P^T A P = D$, $P^T = P^{-1}$

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Result:

A symm $n \times n$ matrix, $\underline{v}, \underline{w}$ eigenvectors with **different** eigenvalue $\Rightarrow \underline{v} \cdot \underline{w} = 0$

Ex 2: (difficult)

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

$$tr(A) = 9 = \lambda_1 + \lambda_2 + \lambda_3$$

i) **Eigenvalues:**

$\lambda = 1$ eigenvalue $\rightarrow$ Null$\begin{pmatrix} 2 & 2 & 2 \\ \cancel{2} & \cancel{2} & \cancel{2} \\ \cancel{2} & \cancel{2} & \cancel{2} \end{pmatrix}$

$m \geq 2$ 2 free var

$\lambda_1 = 1 \quad \lambda_2 = 1 \quad \lambda_3 = 7$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

ii) **Eigenvectors:**

$E_1 = $ Null$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x+y+z=0$, y,z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$E_7 = $ Null$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{v}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\underline{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Downarrow$

Orthogonal diag:

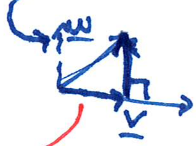
$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix} \quad P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

$$P^T A P = D$$

$(-1,1,0) \cdot (1,1,1) = 0$

$(-1,0,1) \cdot (1,1,1) = 0$

$\underbrace{(-1,1,0)}_{\underline{v}} \cdot \underbrace{(-1,0,1)}_{\underline{w}} = 1 \neq 0$



$$\text{proj}_{\underline{v}}(\underline{w}) = \frac{\underline{w} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \cdot \underline{v} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Replace $\underline{w}$ with $\underline{w} - \text{proj}_{\underline{v}}(\underline{w})$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

② Definiteness of quadratic forms

Defn: A quadratic form is a polynomial fn. where all terms have degree two.

Ex: $q(x,y,z) = 3x^2 + 4xy + 4xz + 3y^2 + 4yz + 3z^2$

$$= (x\ y\ z) \cdot \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$q(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$ metrix form $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

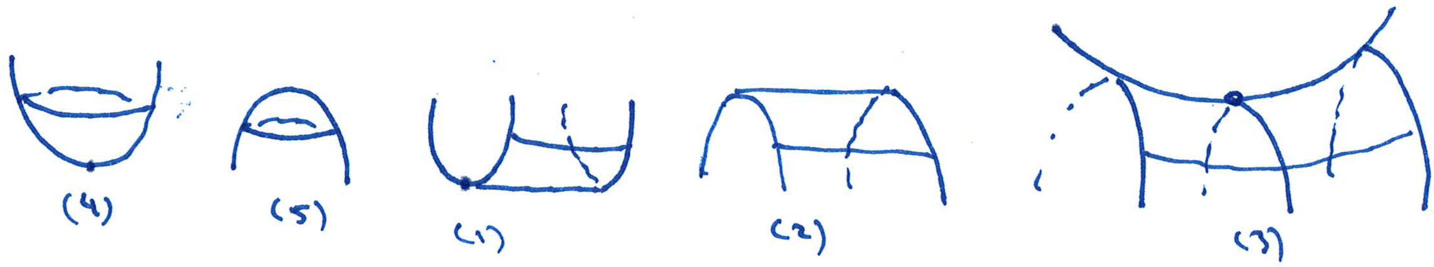
Symmetric 3x3 matrix

Fact: Any quadratic form in n variables $q(\underline{x}) = q(x_1, x_2, \ldots, x_n) = \underline{x}^T A \underline{x}$, where A is a unique symmetric $n \times n$-matrix.

Defn: $q(\underline{x}) = \underline{x}^T A \underline{x}$ quadratic form $\quad q(\underline{0}) = 0$

1) q is positive semidefinite if $q(\underline{x}) \geq 0$ for all $\underline{x}$ $\quad \underline{x} = \underline{0}$ is min
2) " negative semidefinite " $q(\underline{x}) \leq 0$ —"— $\quad \underline{x} = \underline{0}$ is max
3) indefinite otherwise $\quad \underline{x} = \underline{0}$ is saddle pt

4) q is positive definite if $q(\underline{x}) > 0$ —"— $\underline{x} \neq \underline{0}$ $\Rightarrow$ pos. semidefn
5) " negative definite if $q(\underline{x}) < 0$ —"— $\Rightarrow$ neg. semidefn.



Thm
Any quadratic form q in n variables can be written

$$q(\underline{x}) = \lambda_1 u_1^2 + \lambda_2 \cdot u_2^2 + \dots + \lambda_n u_n^2$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the symmetric matrix A and $u_1, u_2, \dots, u_n$ are new coordinates.

Consequence:
- q pos. defn. $\Leftrightarrow \lambda_1, \dots, \lambda_n > 0$
- q neg. defn. $\Leftrightarrow \lambda_1, \dots, \lambda_n < 0$
- q pos. semidefn. $\Leftrightarrow \lambda_1, \dots, \lambda_n \geq 0$
- q neg. semidefn $\Leftrightarrow \lambda_1, \dots, \lambda_n \leq 0$
- q indefinite otherwise

Ex: $q = 3x^2 + 4xy + 4xz + 3y^2 + 4yz + 3z^2 = \underline{1 \cdot u_1^2 + 1 \cdot u_2^2 + 7 \cdot u_3^2}$

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = \lambda_2 = 1 \\ \lambda_3 = 7 \end{matrix}$$

pos. definite

Note: $\underline{\underline{x} = P \cdot \underline{u} \Leftrightarrow P^T \cdot \underline{x} = \underline{u}}$: $\qquad P^T A P = D$

$$\underline{u}^T D \underline{u} = \underline{u}^T (P^T A P) \cdot \underline{u} = \underline{u}^T P^T \cdot A \cdot (P \cdot \underline{u}) = (P\underline{u})^T A \cdot (P\underline{u})$$

$$= \underline{x}^T A \underline{x} = q(\underline{x})$$

$$\Rightarrow q(\underline{x}) = \underline{u}^T D \underline{u} = (u_1\, u_2 \dots u_n) \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$= \underline{\lambda_1 \cdot u_1^2 + \lambda_2 \cdot u_2^2 + \dots + \lambda_n \cdot u_n^2}$$

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = P^T \cdot \underline{x} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \boxed{\begin{aligned} u_1 &= \tfrac{1}{\sqrt{2}}(-x+y) \\ u_2 &= \tfrac{1}{\sqrt{6}}(-x-y+2z) \\ u_3 &= \tfrac{1}{\sqrt{3}}(x+y+z) \end{aligned}}$$

$q(\underline{x}) = \underline{x}^T A \underline{x}$

A symm. $n \times n$-matrix

Method: Principal minors

(a) Leading principal minors

A $n\times n$ matrix

$D_1 = M_{1,1}$
$D_2 = M_{12,12}$
$D_3 = M_{123,123}$
$\vdots$
$D_n = |A|$

Ex: $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ pos. defn.

$D_1 = 3$
$D_2 = 9-4 = 5$
$D_3 = |A| = 3\cdot 5 - 2\cdot 2 + 2(-2) = 7$

Result: A $n\times n$ symmetric matrix

$D_1, D_2, \ldots, D_n > 0 \iff A$ pos. defn.

$D_1 < 0, D_2 > 0, \ldots \iff A$ neg. defn.

Ex: $q(x,y,z) = -x^2 - y^2 - z^2$ neg. defn.

$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $D_1 = -1$, $D_2 = 1$, $D_3 = -1$

(b) All principal minors: Δ_i = any principal minor of order i

Ex: $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$

$\Delta_1 = M_{1,1}, M_{2,2}, M_{3,3} = 3, 3, 3$
$\Delta_2 = M_{12,12}, M_{23,23}, M_{13,13} = 5, 5, 5$
$\Delta_3 = |A| = 7$

Result: A $n\times n$ symm.

i) $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$ for all principal minors $\iff A$ pos. semidefn.

ii) $\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$ for all principal minors $\iff A$ neg. semidefn

iii) all other cases $\iff A$ indefinite

Typical examples of indefinite: $D_2 / \Delta_2 < 0$, $\Delta_4 < 0$, $\Delta_6 < 0, \ldots$
$\Delta_1 > 0$, $\Delta_3 < 0$

c) Reduced rank criterion: RRC

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{pmatrix}$ $|A| = 0$, $\text{rk}A = 2$ $\left.\begin{matrix} D_1 = 2 \\ D_2 = 3 \end{matrix}\right\}$ $D_3 = 0$

Symm. 3x3 matrix

Reduced rank: $\text{rk}\, A < n \Leftrightarrow |A| = 0$

RRC:

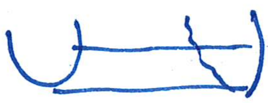
If A is a symmetric $n \times n$ matrix with $\text{rk}\, A = r < n$:

i) $D_1, D_2, \dots, D_r > 0 \Rightarrow A$ positive semidefn.

ii) $D_1 < 0, D_2 > 0, \dots$ up to the rank $\Rightarrow A$ neg. semidefn.

Ex: $q(x,y,z) = 2x^2 + 2xy + 6xz + 2y^2 + 6yz + 6z^2$

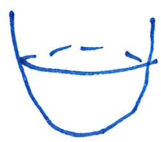
$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{pmatrix}$ $D_1 = 2$, $D_2 = 3$, $D_3 = 0$ $\Rightarrow$



graph of q

pos. semidefn.

$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 7 \end{pmatrix}$ $D_1 = 2$, $D_2 = 3$, $D_3 = 3$ $\Rightarrow$



graph of q

pos. defn.