Plan
1 Eigenvalues and eigenvectors
2 Diagonalization
3 Application: Powers of matrices

Monday: Plenary Session 1
A1-040 16-19
Problems Lecture 1-4

① Eigenvalues and eigenvectors

A $n\times n$ matrix

$(*)$ $A\cdot \underline{v} = \lambda\cdot\underline{v}$ $\quad \begin{cases} \underline{v}: n\text{-vector} \\ \lambda: \text{scalar (no.)} \end{cases}$

$A\underline{v} - \lambda\underline{v} = \underline{0}$

$(A-\lambda I)\underline{v} = \underline{0}$

$(*)$ $\boxed{(A-\lambda I)\underline{v} = \underline{0}}$

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $\quad \lambda I = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots \ \lambda \end{pmatrix}$

Ex: $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$: $A\cdot\underline{x} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+2y \\ 2x+3y \end{pmatrix} = \lambda\underline{v} = \lambda\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

$3x+2y = \lambda x$
$2x+3y = \lambda y$

$3x - \lambda x + 2y = 0$
$2x + 3y - \lambda y = 0$

$\boxed{\begin{aligned} (3-\lambda)x + 2y &= 0 \\ 2x + (3-\lambda)y &= 0 \end{aligned}}$

$\underbrace{\begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}}^{A-\lambda I} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(*)$ homogeneous 2×2 linear system with parameter λ

homog. linear system w/parameter λ

Defn: A $n\times n$-matrix

i) λ is an <u>eigenvalue</u> of A if $A\underline{v} = \lambda\underline{v}$ has non-trivial solutions $\underline{v} \neq \underline{0}$.

ii) $\underline{v}$ is an <u>eigenvector</u> of A with eigenvalue λ if it satisfies $A\underline{v} = \lambda\underline{v}$.

We call $E_\lambda = \{\underline{v} : A\underline{v} = \lambda \underline{v}\}$ the eigenspace of A with eigenvalue λ

all the eigenvectors with eigenvalue λ

Ex: $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$: $A\underline{v} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 5$ eigenvalue $\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in E_5

eigenvector with $\lambda = 5$

$A\underline{v} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \end{pmatrix} = \ ? \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

No.

$\underline{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ not eigenvector

Facts: A $n \times n$-matrix

i) λ eigenvalue of $A \iff |A - \lambda I| = 0$

characteristic eqn.

ii) For each eigenvalue λ, $E_\lambda = \{\underline{v} : \underbrace{A\underline{v} = \lambda\underline{v}}_{(A-\lambda I)\cdot \underline{v} = \underline{0}}\} = \mathrm{Null}(A - \lambda I)$

Ex: $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

i) Eigenvalues: $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 - 2^2 = 0$$

$$9 - 6\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

Eigenvalues of A: $\lambda = 1$ or $\lambda = 5$

n=2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{tr(A)}\lambda + \underbrace{(ad-bc)}_{det(A)} = 0$$

$\boxed{\text{Formula: } \lambda^2 - tr(A)\lambda + |A| = 0}$

ii) Eigenvectors: $A - \lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}$

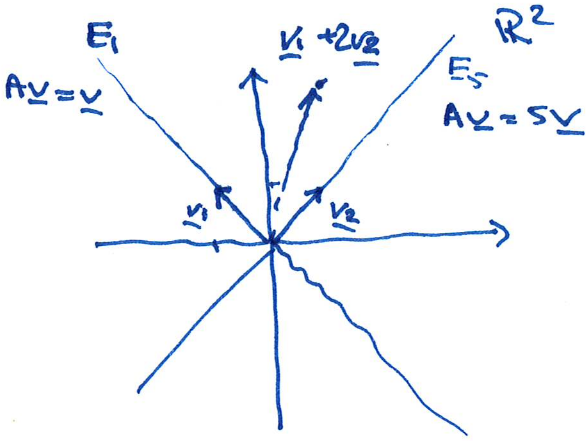
$\lambda = 1$: $\text{Null}(A - I) = \text{Null}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\quad$ $2x+2y=0$, $\cancel{2x+2y=0}$ $\quad$ $x = -y$, y free

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\quad$ $E_1 = \text{span}(\underline{v_1})$ $\quad$ $\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \leftarrow$ Base

$\lambda = 5$: $\text{Null}(A - 5I) = \text{Null}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ $\quad$ $-2x+2y=0$, $\cancel{2x-2y=0}$ $\quad$ $x = y$, y free

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_5 = \text{span}(\underline{v_2})$, $\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow$ Base



$$A(\underline{v_1} + 2\underline{v_2}) = A\underline{v_1} + 2A\underline{v_2} = \underline{v_1} + 2(5\underline{v_2}) = \underline{v_1} + 10\underline{v_2}$$

Ex: $\begin{pmatrix} 4 & 3 \\ -3 & 0 \end{pmatrix}$ $\quad$ $\lambda^2 - 4\lambda + 9 = 0$ $\quad$ $\lambda = \dfrac{4 \pm \sqrt{16-36}}{2}$

no (real) eigenvalues

$\begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}$ $\quad$ $\lambda^2 - 4\lambda + 4 = 0$ $\quad$ $\lambda = \dfrac{4 \pm \sqrt{16-16}}{2} = 2 \pm 0$

$(\lambda - 2)^2 = 0$

$\lambda_1 = \lambda_2 = 2$

(multiplicity 2)

Defn: $\lambda = \lambda_i$ has multiplicity m if $(\lambda - \lambda_i)^m$ is a factor in $|A - \lambda I|$.

Results: A $n\times n$ matrix

i) $|A-\lambda I|=0$ is a polynomial eqn. of order n and it has at most n roots (counted with multiplicity)

ii) if λ is an eigenvalue of multiplicity $m\geq 1$, then

$$1 \leq \underbrace{\dim E_\lambda}_{\text{no. of free variables}} \leq m$$

Ex: $A=\begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$

i) **Eigenvalues:** $|A-\lambda I|=0$

$$\begin{vmatrix} 3-\lambda & 0 & 2 \\ 0 & 5-\lambda & 0 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0 \qquad +(5-\lambda)\cdot\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix}=0$$

$$(5-\lambda)\cdot(\lambda^2-6\lambda+5)=0$$
$$(5-\lambda)\cdot(\lambda-1)(\lambda-5)=0$$
$$-(\lambda-5)(\lambda-1)(\lambda-5)=0$$
$$-(\lambda-5)^2\cdot(\lambda-1)=0$$

$5-\lambda=0$ or $\lambda^2-6\lambda+5=0$

$\lambda_1=5$ $\qquad \lambda_2=5,\ \lambda_3=1$

$\underset{(m=2)}{\lambda_1=\lambda_2=5},\ \underset{(m=1)}{\lambda_3=1}$

ii) **Eigenvectors:**

$E_1 = \text{Null}\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$ $\quad z$ free, $\dim E_1 = 1$

$2x+2z=0 \quad x=-z$

$4y=0 \quad y=0$

z free

$\underline{v}=\begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\qquad \underline{v}_1=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \leftarrow$ Base

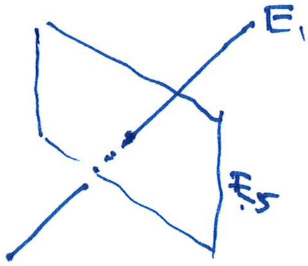
$E_1=\text{span}(\underline{v}_1)$

$$E_5 = \text{Null}\begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \qquad -2x + 2z = 0 \quad \Rightarrow x = z$$

y, z free

$$\underline{v} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \underbrace{\underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Base of } E_5}$$

$$E_5 = \text{Span}(\underline{v_2}, \underline{v_3})$$



② Diagonalization

A $n \times n$ matrix

Defn: A is diagonalizable if there is an invertible matrix P such that

$$P^{-1}AP = D \longleftarrow D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \text{ diagonal matrix}$$

is a diagonal matrix.

Idea: $\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of A $\rightsquigarrow$ $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

$\underline{v_1}, \underline{v_2}, \ldots \underline{v_n}$ eigenvector of A $\rightsquigarrow$ $P = (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})$

$$A \cdot P = A \cdot (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n})$$

$$= (A\underline{v_1} | A\underline{v_2} | \ldots | A\underline{v_n}) = (\lambda_1 \cdot \underline{v_1} | \lambda_2 \cdot \underline{v_2} | \ldots | \lambda_n \cdot \underline{v_n})$$

$$= (\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = P \cdot D$$

$$AP = PD \Rightarrow P^{-1}AP = \cancel{P^{-1}} \cdot \cancel{P}D = D \Rightarrow P^{-1}AP = D$$

Result: A $n \times n$ matrix

A is diagonalizable $\iff$ $\begin{cases} \text{i) } A \text{ has } n \text{ eigenvalues } \lambda_1, \dots, \lambda_n \text{, counted with multiplicity} \\ \text{ii) } A \text{ has } n \text{ linearly independent eigenvectors } \underline{v_1}, \dots, \underline{v_n} \end{cases}$

Assume that $A \cdot \underline{v_i} = \lambda_i v_i$

In that case, $P^{-1}AP = D$ when

$$P = \left(\underline{v_1} \mid \underline{v_2} \mid \dots \mid \underline{v_n} \right), \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$$

Note: ii) $\iff$ $\dim E_\lambda = m$ for any eigenvalue λ of multiplicity m.

In that case, we can collect all base vectors of E_λ for all eigenvalues λ to get a lin. indep. set of eigenvectors.

Ex: $A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}$ not diagonal

i) Eigenvalues: $\underbrace{\lambda_1 = \lambda_2 = 5}_{m=2}$, $\underbrace{\lambda_3 = 1}_{m=1}$

ii) Eigenvectors:

$m=1$: $E_1 = \text{span}(\underline{v_1})$ $\quad \underline{v_1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$m=2$: $E_5 = \text{span}(\underline{v_2}, \underline{v_3})$ $\quad \underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \underline{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P^{-1}AP = D \iff \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A is diagonalizable

Result: A $n\times n$ matrix

i) A symmetric $\Rightarrow$ A is diagonalizable

ii) If A has n distinct eigenvalues, then A is diagonalizable

iii) If A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \qquad tr(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

Sum of diagonal entries in A

3 Powers of matrices

$$\underset{n\times n}{A} \rightsquigarrow A^N \quad (N \geq 0 \text{ big})$$

$$A^N = \underbrace{A \cdot A \cdot \ldots \cdot A}_{N \text{ times}}$$

Note:

Assume that A is diagonalizable, i.e. $P^{-1}AP = D$.

Then $\cancel{P} \cdot \cancel{P^{-1}} AP = P \cdot D$, and $A\cancel{P} \cdot \cancel{P^{-1}} = PD \cdot P^{-1} \Rightarrow \underline{A = PDP^{-1}}$

This means:

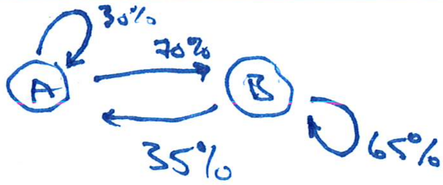
$$A^N = (PDP^{-1})^N = (PD\cancel{P^{-1}}) \cdot (\cancel{P}D\cancel{P^{-1}}) \cdot (\cancel{P}D\cancel{P^{-1}}) \cdot \ldots \cdot (\cancel{P}DP^{-1})$$

$$\boxed{A^N = PD^NP^{-1}}$$

$$A^N = (\underline{v_1}|\underline{v_2}|\ldots|\underline{v_n}) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}^N \cdot (\underline{v_1}|\underline{v_2}|\ldots|\underline{v_n})^{-1}$$

$$= (\underline{v_1}|\underline{v_2}|\ldots|\underline{v_n}) \cdot \begin{pmatrix} \lambda_1^N & & & 0 \\ & \lambda_2^N & & \\ & & \ddots & \\ 0 & & & \lambda_n^N \end{pmatrix} \cdot (\underline{v_1}|\underline{v_2}|\ldots|\underline{v_n})^{-1}$$

Ex: Markov chains



$\underline{v_0} = \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$ initial state $\qquad A = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix}$ transition matrix

$$\underline{v_1} = A \cdot \underline{v_0} = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix} \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix} = \begin{pmatrix} 0.30 \cdot 0.88 + 0.35 \cdot 0.12 \\ 0.70 \cdot 0.88 + 0.65 \cdot 0.12 \end{pmatrix}$$

$$\underline{v_2} = A \cdot \underline{v_1} = A \cdot (A \underline{v_0}) = A^2 \cdot \underline{v_0}$$

$$\vdots$$

$$\underline{v_N} = A^N \cdot \underline{v_0}$$ use diagonalization to compute A^N

$$A = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix}$$

$\mathrm{tr}(A) = 0.30 + 0.65 = 0.95$

$\det(A) = 0.30 \cdot 0.65 - 0.7 \cdot 0.35 = -0.05$

i) Eigenvalues: $\lambda^2 - 0.95\lambda - 0.05 = 0$

$$\lambda = \frac{0.95 \pm \sqrt{0.95^2 - 4(-0.05)}}{2} = \frac{0.95 \pm 1.05}{2} \Rightarrow \underline{\lambda_1 = 1}, \ \underline{\lambda_2 = -0.05}$$

ii) Eigenvectors:

$\lambda = 1$: $E_1 = \mathrm{Null} \begin{pmatrix} -0.7 & 0.35 \\ \cancel{0.7} & \cancel{-0.35} \end{pmatrix}$ $\quad -0.7x + 0.35y = 0 \Rightarrow x = y/2$, y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/2 \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \Rightarrow E_1 = \mathrm{span}(\underline{v_1}) \qquad \underline{v_1} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -0.05}$: $E_{-0.05} = \mathrm{Null} \begin{pmatrix} 0.35 & 0.35 \\ \cancel{0.7} & \cancel{0.7} \end{pmatrix}$ $\quad 0.35x + 0.35y = 0 \Rightarrow x = -y$, y free

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow E_{0.05} = \mathrm{span}(\underline{v_2}) \qquad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & -0.05 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/2 & -1 \\ 1 & 1 \end{pmatrix} \qquad P^{-1} = \frac{1}{1/2+1}\begin{pmatrix} 1 & 1 \\ -1 & 1/2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 & 1 \\ -1 & 1/2 \end{pmatrix}$$

$$\Downarrow$$

$$A^N = P \cdot D^N P^{-1} = \begin{pmatrix} 1/2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \overbrace{1^N}^{=1} & 0 \\ 0 & (-0.05)^N \end{pmatrix} \cdot \frac{2}{3}\begin{pmatrix} 1 & 1 \\ -1 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -(-0.05)^N \\ 1 & (-0.05)^N \end{pmatrix} \cdot \frac{2}{3}\begin{pmatrix} 1 & 1 \\ -1 & 1/2 \end{pmatrix}$$

$$= \frac{2}{3}\begin{pmatrix} 1/2 + (-0.05)^N & 1/2 - 1/2(-0.05)^N \\ 1 - (-0.05)^N & 1 + 1/2(-0.05)^N \end{pmatrix}$$

$$= \frac{2}{3}\begin{pmatrix} 1/2 & 1/2 \\ 1 & 1 \end{pmatrix} + \frac{2}{3}(-0.05)^N \cdot \begin{pmatrix} 1 & -1/2 \\ -1 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} + (-0.05)^N \cdot \begin{pmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{pmatrix} \simeq \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}$$

$$\underline{v}_N = A^N \cdot \underline{v}_0 \simeq \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}\begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix} = 0.88\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} + 0.12\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}}$$

long term equilibrium state when $N \to \infty$