Plan

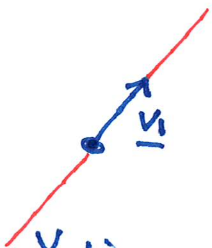
1 Vector spaces and span

$L = \{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_m}\}$ n-vectors

$$V = \mathrm{span}(L) = \{ \underbrace{c_1 \underline{v_1} + c_2 \underline{v_2} + \ldots + c_m \cdot \underline{v_m}}_{\text{linear combination of the vectors } \underline{v_1}, \ldots, \underline{v_m}} : c_1, \ldots, c_m \text{ are numbers} \}$$

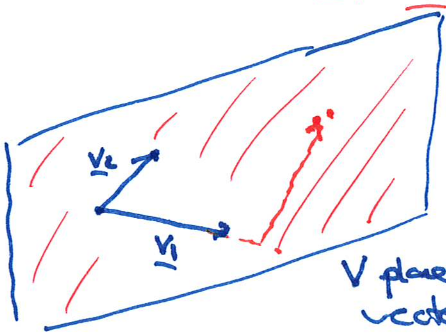
Ex1

$\underline{m=1}$:



V line
vector space of dimension 1

$\underline{m=2}$:



V plane
vector space of dim. 2

Defn: A subset V of $\mathbb{R}^n$ is a <u>vector space</u> if $V = \mathrm{span}(L)$, where $L = \{\underline{v_1}, \ldots, \underline{v_m}\}$ is a collection of n-vectors.

Ex:

$$\begin{aligned} x+y+z+w &= 5 \\ x-y+3z-w &= 5 \\ x+3y-z+3w &= 5 \end{aligned} \quad (*)$$

3×4 lin. system

$$\begin{aligned} x+y+z+w &= 0 \\ x-y+3z-w &= 0 \\ x+3y-z+3w &= 0 \end{aligned} \quad (**)$$

3×4 homogeneous lin. system

(*)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & -1 & 3 & -1 & 5 \\ 1 & 3 & -1 & 3 & 5 \end{array}\right) \begin{array}{l} \\ -1 \\ -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 2 & -2 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \\ 1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\underbrace{x, y}_{\text{basic}}$ $\underbrace{z, w}_{\text{free}}$

$$\begin{aligned} x + y + z + w &= 5 \\ -2y + 2z - 2w &= 0 \end{aligned}$$

$$y - z + w = 0 \Rightarrow y = \underline{z - w}$$

$$x + (z - \cancel{w}) + z + \cancel{w} = 5 \Rightarrow x = \underline{5 - 2z}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5-2z \\ z-w \\ z \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z \cdot \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\underline{w_1}} + w \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\underline{w_2}}$$

$$\begin{cases} z = s \\ w = t \end{cases}$$

$$= \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \text{span}(\underline{w_1}, \underline{w_2})$$

(**)

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} \underline{x} + y + z + w &= 0 \\ -2\underline{y} + 2z - 2w &= 0 \end{aligned}$$

$$y = \underline{z - w}$$

$$x = \underline{-2z}$$

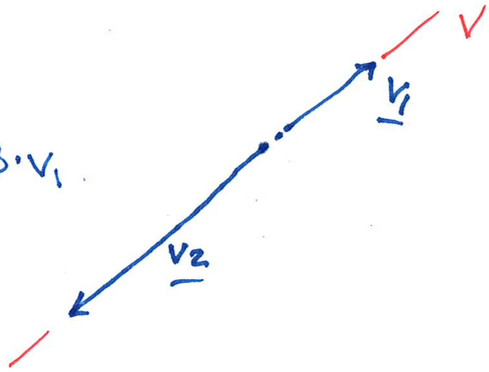
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = z \cdot \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{\underline{w_1}} + w \cdot \underbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{\underline{w_2}}$$

$$= \text{span}(\underline{w_1}, \underline{w_2})$$

(2) Linear independence and dimension

A subset V of $\mathbb{R}^n$ is a vector space if $V = \text{span}(L)$, where $L = \{\underline{v_1}, \ldots, \underline{v_m}\}$ is a collection of n-vectors.

If the vectors in L is a minimal spanning set of vectors, we call L a base of V



Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix} = -3 \cdot v_1$

$$\begin{aligned} V &= \text{span}(\underline{v_1}, \underline{v_2}) \\ &= \{ c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} \} \\ &= \{ c_1 \cdot \underline{v_1} + c_2 \cdot (-3\underline{v_1}) \} \\ &= \{ (c_1 - 3c_2) \cdot \underline{v_1} \} \\ &= \text{span}(\underline{v_1}) \end{aligned}$$

Linear independence:

A set $L = \{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_m}\}$ of n-vectors are linearly dependent if at least one of the vectors in L is a linear combination of the others, and linear independent otherwise.

Ex: $\underline{w_1} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $\underline{w_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{w_2} = c \cdot \underline{w_1}$? NO.
$\underline{w_1} = c \cdot \underline{w_2}$? NO

Linearly independent.

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix}1\\3\\-1\end{pmatrix}$ $\underline{v}_4 = \begin{pmatrix}1\\-1\\3\end{pmatrix}$ $\longrightarrow A = \begin{pmatrix}1&1&1&1\\1&-1&3&-1\\1&3&-1&3\end{pmatrix}$

$$\downarrow$$

$$\begin{pmatrix}1&1&1&1\\0&-2&2&-2\\0&0&0&0\end{pmatrix}$$

echelon form

① $\{\underline{v}_1, \underline{v}_2\}$ are linearly independent

② $\underline{v}_3, \underline{v}_4$ are linear combinations of $\underline{v}_1, \underline{v}_2$ $\Longleftarrow$

General facts:

① Let $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m$ be n-vectors

$$\left.\begin{array}{c} x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + \ldots + x_m \cdot \underline{v}_m = \underline{0} \\ \Updownarrow \\ A \cdot \underline{x} = \underline{0} \end{array}\right\}$$

- one unique solution ($\underline{x} = \underline{0}$): $\{\underline{v}_1, \ldots, \underline{v}_m\}$ $\boxed{\text{linearly independent}}$
- infinitely many solutions (at least one degree of freedom) $\boxed{\text{linearly dependent}}$

Ex: $x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\-1\\3\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\3\\-1\end{pmatrix} + x_4 \cdot \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ x_1 - x_2 + 3x_3 - x_4 \\ x_1 + 3x_2 - x_3 + 3x_4 \end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}1&1&1&1\\1&-1&3&-1\\1&3&-1&3\end{pmatrix}\underline{x} = \underline{0}$$

② linearly dependent

$\Updownarrow$

$\boxed{x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots + x_m\underline{v}_m = \underline{0}}$

$$\left.\begin{array}{lcl} \underline{v}_1 = c_2\underline{v}_2 + c_3\underline{v}_3 + \ldots + c_m\underline{v}_m & \Leftrightarrow & -1 \cdot \underline{v}_1 + c_2\underline{v}_2 + \ldots + c_m\underline{v}_m = \underline{0} \\ \text{or} & & \\ \underline{v}_2 = c_1\underline{v}_1 + c_3\underline{v}_3 + \ldots + c_m\underline{v}_m & & c_1\underline{v}_1 + (-1)\underline{v}_2 + \ldots \quad = \underline{0} \\ \text{or} & & \vdots \\ \vdots & & \end{array}\right\}$$

nontrivial solutions of

$$-2\underline{v_1} + \underline{v_2} + \underline{v_3} = \underline{0}$$
$$-\underline{v_2} + \underline{v_4} = \underline{0}$$

$\cancel{x}_1 + y + z + w = 0$

$-2y + 2z - 2w = 0$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2z \\ z-w \\ z \\ w \end{pmatrix} = z \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$\Updownarrow$

$$\boxed{\begin{aligned} \underline{v_3} &= 2\underline{v_1} - \underline{v_2} \\ \underline{v_4} &= \underline{v_2} \end{aligned}}$$

Summary: Linear independence

$\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}$ $\rightsquigarrow$ $A = (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_m})$

n-vectors — $n \times m$ matrix

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}\}$ linearly independent $\Leftrightarrow$ $A\underline{x} = \underline{0}$ has only the trivial solution $\underline{x} = \underline{0}$ (no free variables)

— " — linearly dependent $\Leftrightarrow$ $A\underline{x} = \underline{0}$ has non-trivial solutions (at least one free var.)

and

the vectors corresponding to pivot position is the max. subset of linearly independent vectors

Defn:

A basis of a vector space $V = \text{span}(\underline{v_1}, \dots, \underline{v_m})$ is a subset of the vectors $\{\underline{v_1}, \dots, \underline{v_m}\}$ that are linearly independent and span V, i.e. a minimal set of vectors that span V.

The dimension $\dim V$ of V is the number of vectors in a basis of V.

Ex:

$V = \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}, \underline{v_4})$

Basis: $\{\underline{v_1}, \underline{v_2}\}$

$\dim V = \underline{\underline{2}}$

Important vector spaces

$\underset{\substack{n\times m\\ \text{matrix}}}{A} = \left(\underline{v_1} \mid \underline{v_2} \mid \cdots \mid \underline{v_m} \right)$

i) $\underline{\text{Null}(A)} = \{ \underline{x} : A\underline{x} = \underline{0} \}$

null-space of A

Results:

$\dim \text{Null}(A) = m - \text{rk}(A)$

$A \cdot \underline{x} = \underline{0}$ has solution

$t_1 \cdot \underline{w_1} + t_2 \cdot \underline{w_2} + \cdots + t_r \cdot \underline{w_r} \leftarrow \text{span}(\underline{w_1}, \underline{w_2}, \ldots, \underline{w_r})$

where $t_1, t_2, \ldots, t_r$ are the free variables (r = # degrees of freedom)

$r = m - \text{rk}(A)$

Base: $\{\underline{w_1}, \underline{w_2}, \ldots, \underline{w_r}\}$

Ex: $A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$ $\begin{matrix} \\ -2 \\ -3 \end{matrix}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & 3 & -6 \end{pmatrix}$ $\begin{matrix} \\ \\ -1 \end{matrix}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$

$\underline{\text{Null}(A)}$: $\dim \text{Null}(A) = 3 - \text{rk}(A) = 3 - 2 = \underline{1}$.

Null(A) is a line in $\mathbb{R}^3$

Base: $\underline{w_1} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

$\underline{x} - y + 3z = 0$
$\underline{3y} - 6z = 0$

$\left.\begin{aligned} x &= -z \\ y &= 2z \\ z &= z \end{aligned}\right\} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 2z \\ z \end{pmatrix} = z \cdot \underline{\underline{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}}$

ii) $\text{Col}(A) = \text{span}(\underline{v_1}, \underline{v_2}, \ldots, \underline{v_m})$ $\qquad A = (\underline{v_1} \mid \underline{v_2} \mid \ldots \mid \underline{v_m})$

~~dim Col(A) =~~

Results:

$\dim \text{Col}(A) = \text{rk}(A)$

Base: column vectors that correspond to pivots

Ex: $A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$ $\qquad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

$\downarrow$

$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$

$\{\underline{v_1}, \underline{v_2}\}$ are linearly independent

$\underline{v_3}$ is a linear comb of $\underline{v_1}, \underline{v_2}$

$\Downarrow$

Base of $\text{Col}(A)$: $\{\underline{v_1}, \underline{v_2}\}$

$\dim \text{Col}(A) = 2$

Null(A)

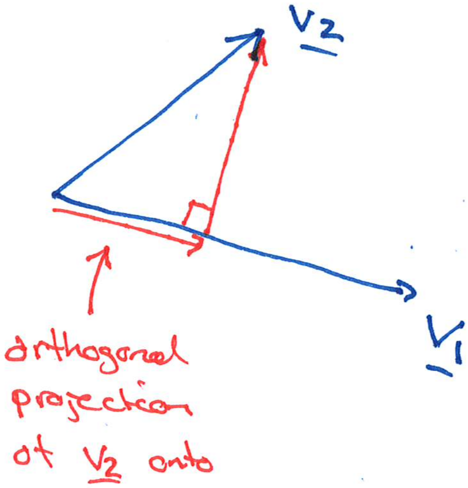
$$\boxed{A\underline{x} = \underline{0}: \quad \underline{x} = z \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$$

$$\left.\begin{aligned} -1 \cdot \underline{v_1} + 2\underline{v_2} + \underline{v_3} &= \underline{0} \\ \underline{v_3} &= \underline{v_1} - 2\underline{v_2} \end{aligned}\right\} \quad \text{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \text{span}(\underline{v_1}, \underline{v_2})$$

$\Downarrow$

Col(A) is a plane in $\mathbb{R}^3$

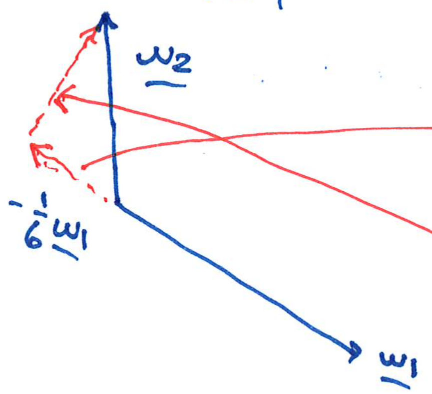
③ Inner products and projections



$$\text{proj}_{\underline{v}_1}(\underline{v}_2) = k \cdot \underline{v}_1 \quad \text{where} \quad k = \frac{\underline{v}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}$$

Ex:

$$\underline{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \underline{w}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\underline{w}_2 \cdot \underline{w}_1 = -2 \cdot 0 + 1 \cdot (-1) + 1 \cdot 0 + 0 \cdot 1 = -1$$

$$\underline{w}_1 \cdot \underline{w}_1 = (-2)^2 + 1^2 + 1^2 + 0^2 = 6$$

$$\text{proj}_{\underline{w}_1}(\underline{w}_2) = \frac{\underline{w}_2 \cdot \underline{w}_1}{\underline{w}_1 \cdot \underline{w}_1} \cdot \underline{w}_1 = \frac{-1}{6} \underline{w}_1$$

$$= -\frac{1}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/3 \\ -1/6 \\ -1/6 \\ 0 \end{pmatrix}}}$$

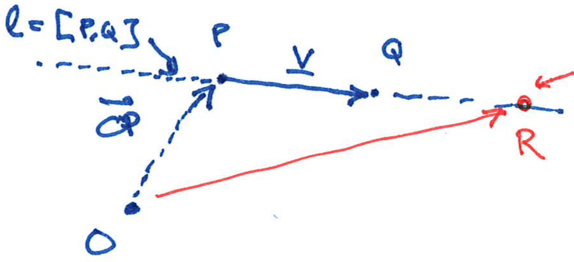
$$\underline{w}_2 - \left(-\frac{1}{6}\underline{w}_1\right) = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ -1/6 \\ -1/6 \\ 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -1/3 \\ -5/6 \\ 1/6 \\ 1 \end{pmatrix}}}$$

Parametrization of the line $[P,Q]$ through two points $P \neq Q$ in $\mathbb{R}^n$:

$$\left.\begin{aligned} P &= (p_1, p_2, \ldots, p_n) \\ Q &= (q_1, q_2, \ldots, q_n) \end{aligned}\right\} \quad \underline{v} = \vec{PQ} = (q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n)$$

$$\begin{aligned} \vec{OP} &= (p_1 - 0, p_2 - 0, \ldots, p_n - 0) \\ &= (p_1, p_2, \ldots, p_n) \end{aligned}$$



$R = (x_1, x_2, \ldots, x_n)$ is any pt on $\ell = [P,Q]$, the line through P and Q:

$$\begin{aligned} \vec{OR} &= \vec{OP} + \vec{PR} \\ &= \vec{OP} + t \cdot \vec{PQ} \qquad \underline{\text{for some scalar } t} \\ &= (p_1, p_2, \ldots, p_n) \\ &\quad + t(q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n) \end{aligned}$$

$\Rightarrow$

Parametrization of the line through P and Q: $\rightarrow$ $\underline{\underline{R = (p_1, p_2, \ldots, p_n) + t \cdot (q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n)}}$

__Ex:__

$$\left.\begin{aligned} P &= (1,2,1) & \vec{PQ} &= (2,-3,-1) \\ Q &= (3,-1,0) & \vec{OP} &= (1,2,1) \end{aligned}\right\} \quad \begin{aligned} (x,y,z) &= (1,2,1) + t(2,-3,-1) \\ &= \underline{\underline{(1+2t, 2-3t, 1-t)}} \end{aligned}$$

is a parametrization of the line through P and Q in $\mathbb{R}^3$.

Row Space:

$$\underset{\substack{m \times n \\ \text{matrix}}}{A} = \begin{pmatrix} \underline{v_1} \\ \hline \vdots \\ \hline \underline{v_m} \end{pmatrix}$$

We consider the m rows of A to be vectors in $\mathbb{R}^n$, which we may call $\underline{v_1}, \underline{v_2}, \dots, \underline{v_m}$ as shown to the left.

Then we define:

$$\boxed{\text{Row}(A) = \text{span}(\underline{v_1}, \dots, \underline{v_m}) \text{ in } \mathbb{R}^n}$$

(rowspace of A is the span of the row vectors of A)

If $A \to \dots \to E$ is a sequence of elementary row operations, and E is an echelon form, we have:

i) $\boxed{\text{Row}(A) = \text{Row}(E)}$

(the row space does not change under elementary row operations)

ii) $\boxed{\dim \text{Row}(E) = \text{rk}(A)}$

(the number of pivot positions in A, or E), and <u>the non-zero rows</u> in E form a <u>base</u> of Row(A) = Row(E).

<u>Ex:</u> $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 3 \\ 2 & 3 & 0 & 4 \end{pmatrix}$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

* Row(A) = Row(E)

* $\dim \text{Row}(A) = \underline{\underline{2}}$ $(= \text{rk} A)$

* <u>Base</u>: $\underline{v_1} = (1,1,1,1)$, $\underline{v_2} = (0,1,-2,2)$ $\left(\text{or } \underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \; \underline{v_2} = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix}\right)$