Plan

1 Linear systems and their geometry
2 Gaussian elimination
3 Rank of a matrix

Reading:
[EJ Ch. I
(and Forkloo's Lecture I)

About GRA 6035 Mathematics:

Lectures: Fri 08-10 (A1-040) Part 1
Fri 11-12 (Zoom) Part 2

Notes
Video (available 1 week)
Problem sets

Plenary / TA sessions: Mon 17-19 (TA session) - classrooms + Zoom
Mon 17-20 (Plenary session) - A1040 (4x)

Textbook: [E] Eriksen, Gradudate Mathematics

Solutions to problems - H's L.

Exams: Midterm (Oct, 1h MC exam) 20%
Final (Dec, 3h wr. exam) 80%

exam plan to come soon

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About ELE3781 Mathematics elective:

As above, but: * Extra lectures A-C (Wed.) + pb. sessions
* Midterm: Home exam instead of MC. (Python)

Corrections for First Edition, First Printing (2021)

Important corrections

On p. 19, last line: The condition all entries under a pivot are zero should be replaced by the condition a pivot in a lower row is further to the right than a pivot in a higher row.
Comment: In many cases, the conditions are equivalent. But there are examples where this is not the case, such as the following matrix (which is **not** in echelon form):

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

On p. 75, in the proof of Prop. 3.11: The first sentence in the proof should be replaced by the following text: Let r be the rank of A, and write $\mathrm{Row}(A)$ for the row space of A, the vector space spanned by the row vectors of A. Since the row space is preserved under elementary row operations, and the row space of an echelon form has the non-zero rows as a base, it follows that $\dim \mathrm{Row}(A) = r$. We can therefore find r linearly independent rows in A, and write I for any choice of r linearly independent rows, and J for the columns with pivot positions. In the second last line of the proof, the text with s pivot positions should be replaced by with s linearly independent rows.
Comment: The original text give a correct argument in many cases, but not all. Note that elementary row operations may change which rows are linearly independent, for instance when two rows are interchanged.

On p. 99, first line of third paragraph: In the definition, the words irreducible and primitive should be interchanged.

Minor corrections

On p. 11, second line of first paragraph: The word v//tgcqaribles should be variables.
On p. 29, displayed line below the middle: In the set V of solutions, (x, y, z) should be (x, y, z, w).
On p. 40, in the proof of Prop. 2.1: The text formulas 2.1 - 2.1 should be formulas on the previous page.
On p. 96, first line of Section 4.4: The text positive integer should be a positive integer.
On p. 143, Problem 5.1: Part e) and f) should be omitted, as they are identical to Problem 5.2.

List of corrections to the textbook [E] so far; if you find things that you think are wrong, please tell me and I will update this list (link from web page).

1 Linear systems and their geometry

Ex:

$$\boxed{\begin{aligned} x+y+z+w &= 6 \\ x-y+z &= 7 \\ 2x+3y-z+w &= 13 \end{aligned}}$$

3x4 linear system

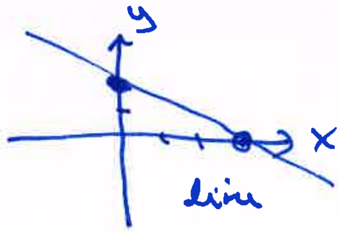
Solutions: all (x,y,z,w) that solves all 3 eqn's.

Defn: A *linear equation* in n variables $x_1, x_2, \ldots, x_n$ is an eqn. of the form

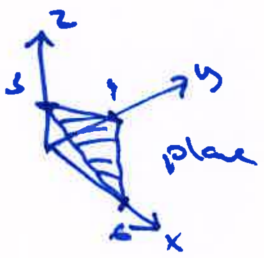
$$\boxed{a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n = b}$$

where $a_1, a_2, \ldots, a_n, b$ are given numbers.
It is called *degenerate* if $a_1 = a_2 = \ldots = a_n = 0$, otherwise it is *non-degenerate*.

Geometry: $2x+3y=6$ (ex. in two vars)



$2x+3y+4z=12$ (ex. in three var's)



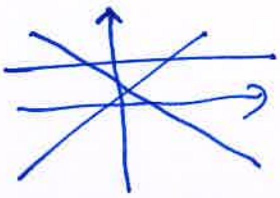
In general, the solutions of any linear eqn. (non-degenerate) is called a *hyperplane*.

Defn. An $m \times n$ linear system in the variables $x_1, \ldots, x_n$ is a system of eqn's that can be written as

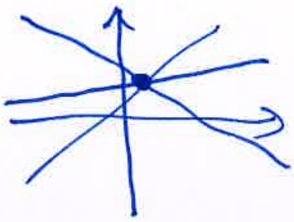
$$m \left\{ \boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}} \right.$$

where a_{ij}, b_i are given numbers.

Ex: $(n=2)$



3×2 system with no sol's



3×2 system with one solution

Results:

① Linear property: Let H be a hyperplane (the solutions of one non-degenerate linear eqn's). If $P \neq Q$ in H, then $[P, Q]$ (the straight line going thr. P and Q) lies inside H.

② Any $m \times n$ linear system has either

i) one solution
ii) infinitely many solutions } consistent
iii) no solutions } inconsistent

② Gaussian elimination

- general method for solving linear systems

Method:

① Start with $m \times n$ linear system in standard form:

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}}$$

② Write down the augmented matrix of the system

$$\begin{array}{c} \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ x_1 & x_2 & & x_n \end{array} \end{array}$$

③ Use elementary row operations to reduce the matrix to echelon form

Elementary row op:
i) Switch two rows
ii) Multiply a row with $c \neq 0$
iii) Add a multiple of one row to another row

Echelon form
i) zero rows should be in the bottom of the matrix
ii) a pivot (the first non-zero entry in a row) in a lower row is further to the right than a pivot in a higher row.

Ex:

$$\begin{pmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form

4. Use back substitution to find the solutions
(start from the last eqn, solve it for the variable in pivot position, move the next last eqn., substitute the variable you solved for, etc.)

Facts: * Any matrix is row equivalent with an echelon form (you can get from one to the other using elementary row operations)
* An echelon form is not unique, but the pivot positions are
* the pivot positions determine the number of solutions:

pivot in the last column $\Leftrightarrow$ no solutions
otherwise:
pivots in all variable col's $\Leftrightarrow$ one solution
at least one variable col. without pivot $\Leftrightarrow$ infinitely many solutions

For consistent systems:
no pivot in a variable col.: variable is free
a pivot in a variable col.: variable is basic (we can solve for it)

Ex:
$$x+y+z+w=7$$
$$x-y+2z+3w=13$$
$$2x+3y-w=5$$

make these zero

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 1 & -1 & 2 & 3 & 13 \\ 2 & 3 & 0 & -1 & 5 \end{array}\right) \begin{array}{l} -1, -2 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 1 & -2 & -3 & -9 \end{array}\right) \begin{array}{l} 1/2 \end{array}$$

get zero

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & -2 & -3 & -9 \\ 0 & -2 & 1 & 2 & 6 \end{array}\right) \begin{array}{l} 2 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & 1 & -2 & -3 & -9 \\ 0 & 0 & -3 & -4 & -12 \end{array}\right)$$

x y z w
echelon form

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 0 & -3/2 & -2 & -6 \end{array}\right) \cdot 2$$

echelon form

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 0 & -2 & 1 & 2 & 6 \\ 0 & 0 & -3 & -4 & -12 \end{array}\right)$$

echelon form

$$x+y+z+w=7$$
$$y-2z-3w=-9$$
$$-3z-4w=-12$$

(3) $\dfrac{-3z}{-3} = \dfrac{4w-12}{-3} \Rightarrow z = -\frac{4}{3}w+4$

(2) $y = 2z+3w-9$

$= 2\left(-\frac{4}{3}w+4\right)+3w-9 \Rightarrow y = \frac{1}{3}w-1$

(1) $x = 7-y-z-w$ $\qquad x = 4$

$= 7-\left(\frac{1}{3}w-1\right)-\left(-\frac{4}{3}w+4\right)-w$

Solutions:

$(x,y,z,w) =$

$\left(4, \frac{1}{3}w-1, -\frac{4}{3}w+4, w\right)$

where w is free

infinitely many solutions, one degree of freedom

x, y, z : basic var
w : free

(3) Rank of a matrix

Defn. The rank of an $m \times n$-matrix A is the number of pivot positions in an echelon form that is row equivalent with A.

We write $\text{rk}(A)$ for the rank of A.
Its value is among the integers $0, 1, 2, \ldots, \min(m,n)$.

Ex:
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 4 & 7 \end{pmatrix} \begin{matrix} \\ ]\,{-1} \\ ]\,{-1} \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{pmatrix}]\,2 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \text{rk}(A) = 2$ two pivot positions

Thm: Consider an $m \times n$ linear system with coefficient matrix A and augmented matrix $(A|\underline{b})$. Then we have:

i) If $\text{rk}(A) < \text{rk}(A|\underline{b})$, then the system is inconsistent (no solutions)

ii) Otherwise, it has (consistent case)
- one solution $\Leftrightarrow \text{rk}(A) = n$
- inf. many solutions $\Leftrightarrow \text{rk}(A) < n$

Note that if a linear system is consistent, and we put
$$V = \{\text{all solutions}\} \subseteq \mathbb{R}^n \leftarrow \text{n-dimensional space}$$
then $\boxed{\dim V = n - \text{rk}(A)}$.

Plan

1. More examples of linear systems
2. Homogeneous linear systems

① More examples of linear systems

Ex:

$$\boxed{\begin{aligned} x+y+z+w &= 5 \\ 2x-y+3z-w &= 2 \\ x+4y+3w &= 12 \end{aligned}}$$

3×4 linear system

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -1 & 3 & -1 & 2 \\ 1 & 4 & 0 & 3 & 12 \end{array}\right) \begin{matrix} -2 \\ -1 \end{matrix} \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 3 & -1 & 2 & 7 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right)$$

echelon form

Infinitely many solutions, one free variable

x, y, w: basic

z: free

$$\begin{aligned} x+y+z+w &= 5 \\ -3y+z-3w &= -8 \\ -w &= -1 \end{aligned}$$

$-3y = -z+3\cdot(1)-8 = -z-5$

$w = 1$

$x = -y-z-w+5$
$= -\left(\frac{1}{3}z+\frac{5}{3}\right)-z-1+5 = \frac{7}{3}-\frac{4}{3}z$

$y = \frac{-z-5}{-3} = \frac{1}{3}z+\frac{5}{3}$

Solutions: $(x,y,z,w) = \left(\frac{7}{3}-\frac{4}{3}z,\ \frac{z}{3}+\frac{5}{3},\ z,\ 1\right)$ with z free

$= \left(\frac{7}{3}-\frac{4}{3}t,\ \frac{1}{3}t+\frac{5}{3},\ t,\ 1\right)$ t parameter

Gauss-Jordan elimination:

Defn: A reduced echelon form is an echelon form such that:
i) each pivot is 1
ii) all entries over a pivot are zero

The reduced echelon form is unique

Ex: $V = \{ (x, y, z, w) = (\frac{7}{3}t - \frac{4}{3}, \frac{1}{3}t + \frac{5}{3}, t, 1) : t \}$

the set of solutions in the example,
a line in 4-dimensional space ($\mathbb{R}^4$)
($\dim V = n - rk(A) = 4 - 3 = 1 =$ # free variables)

(2) Homogeneous systems:

A linear system where all the constant terms $b_1 = b_2 = \dots = b_m$

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}} \Rightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right)$$

Facts: i) homogeneous systems are always consistent

ii) pivots in all var. cols. $\Leftrightarrow$ one solution $\underline{x} = (0, 0, \dots)$ zero sol — only trivial sol
$rk(A) = n$

at least one var. col. without pivot $\Leftrightarrow$ inf. many solutions — there are non-trivial solutions
$rk(A) < n$

Ex:
$$\boxed{\begin{aligned} x + 2y - z &= 0 \\ 2x + y + 3z &= 0 \\ 3x + 3y + z &= 0 \end{aligned}}$$

Solutions: One solution $(x, y, z) = (0, 0, 0)$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} \begin{matrix} -2 \\ -3 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & -3 & 4 \end{pmatrix} -1 \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

Back substitution:

$$\begin{aligned} x + 2y - z &= 0 \\ -3y + 5z &= 0 \\ -z &= 0 \end{aligned} \qquad \begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$$

Ex:

$$\boxed{\begin{aligned} x + 2y - z &= 0 \\ 2x + y + 3z &= 0 \\ 3x + 3y + 2z &= 0 \end{aligned}} \qquad \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ 3 & 3 & 2 \end{pmatrix} \to \dots \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} x + 2y - z &= 0 \\ -3y + 5z &= 0 \end{aligned}}$$

$$\frac{-3y}{-3} = \frac{-5z}{-3} \qquad y = \frac{5}{3}z$$

$$x = -2\left(\frac{5}{3}z\right) + z = -\frac{7}{3}z$$

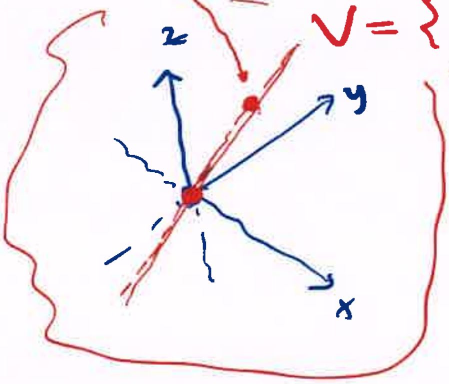
$$(x, y, z) = \left(-\frac{7}{3}z, \frac{5}{3}z, z\right) = \left(-\frac{7}{3}t, \frac{5}{3}t, t\right)$$

1-dim line

(z free)

$\underset{t=3}{\to}$ $(-7, 5, 3)$ is our non-trivial solution

$$V = \left\{\left(-\frac{7}{3}t, \frac{5}{3}t, t\right)\right\}.$$



Key Problems

Problem 1.
Solve the linear systems with the following augmented matrices:

a) $\left(\begin{array}{ccc|c} 1 & 3 & 4 & 11 \\ 5 & 1 & 8 & 15 \\ 4 & 5 & 9 & 23 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 4 & 5 & 11 & 23 \\ 2 & -1 & 3 & 3 \\ 3 & 2 & 7 & 12 \end{array}\right)$ c) $\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 4 & 8 \\ 1 & 3 & 1 & 5 & 18 & 28 \\ 2 & 4 & 2 & 9 & 31 & 48 \end{array}\right)$

Problem 2.
Find the rank of the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 3 & 0 & 2 & 3 & 1 \\ 3 & 6 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 \end{pmatrix}$$

Problem 3.
We consider the following linear system. Find all solutions that satisfies $x + w = y + z$:

$$\begin{array}{ccccccccc} x & + & y & + & 2z & + & 4w & = & 6 \\ x & + & 2y & + & 4z & - & 2w & = & 9 \\ x & + & 3y & + & 9z & + & 7w & = & 24 \end{array}$$

Problem 4.
We consider the homogeneous linear system with the following coefficient matrix:

$$A = \begin{pmatrix} 6 & 6 & 3 & 6 \\ \color{red}{5} & 5 & -1 & 4 \\ 8 & 7 & 7 & 5 \end{pmatrix}$$

Describe the set of solutions geometrically. How many degrees of freedom are there? Does this change if we change the red coefficient in the second row?

Exercise problems

Problems from the textbook: [E] 1.1 - 1.17