

Solution: Key problems — Lecture 9

$$1. a) f'(x) = \frac{p \cdot a}{1+ax} + \frac{(-q)b}{1-bx} = \frac{pa(1-bx) - bq(1+ax)}{(1+ax)(1-bx)}$$

$$= \frac{ap - bq - ab(px+q)x}{(1+ax)(1-bx)} = \frac{ap - bq - abx}{(1+ax)(1-bx)} = 0$$

$$\begin{aligned} ap - bq &= abx \\ x &= \frac{ap - bq}{ab} > 0 \quad \text{since } ap - bq > 0 \end{aligned}$$

$$f''(x) = \frac{-pa^2}{(1+ax)^2} + \frac{-qb^2}{(1-bx)^2} < 0 \Rightarrow f \text{ concave}$$

$$x^* = \frac{ap - bq}{ab} \quad \text{max point}$$

$$b) \left. \begin{array}{l} a=2, b=1 \\ p=0.40, q=0.60 \end{array} \right\} x^* = \frac{2 \cdot 0.4 - 1 \cdot 0.6}{2 \cdot 1}$$

$$= \frac{0.2}{2} = \underline{\underline{0.10}} \quad \text{max point}$$

$$f^* = f(0.10) = 0.40 \ln(1+2 \cdot 0.10) + 0.60 \cdot \ln(1-0.10)$$

$$= 0.40 \ln(1.2) + 0.60 \cdot \ln(0.90)$$

$$\approx \underline{\underline{0.0097}} \quad \text{max value}$$

$$c) \frac{df^*(p)}{dp} = \frac{\partial f}{\partial p} (x=x^*) \quad \leftarrow \frac{df}{dp} = \ln(1+ax) - \ln(1-bx)$$

since $q=1-p$

$$= \ln\left(1 + a \cdot \frac{ap - bq}{ab}\right) - \ln\left(1 - b \cdot \frac{ap - bq}{ab}\right)$$

$$= \ln\left(1 + \frac{ap - bq}{b}\right) - \ln\left(1 - \frac{ap - bq}{a}\right)$$

$$= \ln(1 + 2 \cdot 0.10) - \ln(1 - 1 \cdot 0.10) = \ln(1.2) - \ln(0.9)$$

$$\approx 0.2877$$

$$\left. \begin{array}{l} a=2 \\ b=1 \\ p=0.40 \\ x^*=0.10 \end{array} \right\}$$

Estimate for $p=0.43$:

$$f^*(0.43) \approx f^*(0.40) + 0.03 \cdot \frac{df}{dp} \approx 0.0097 + 0.03 \cdot 0.2877$$

$$= \underline{\underline{0.0183}}$$

2. max $f(x,y,z) = 4x^3 - 2y^3 + z^3$ when $x^3 + y^3 + z^3 \leq 8$

a) $L = 4x^3 - 2y^3 + z^3 - \lambda(x^3 + y^3 + z^3)$

FOC:

$$\begin{aligned} L'_x &= 12x^2 - \lambda \cdot 3x^2 = 0 & 3x^2(4-\lambda) &= 0 & x=0 & \text{or } \lambda=4 \\ L'_y &= -6y^2 - \lambda \cdot 3y^2 = 0 & 3y^2(-2-\lambda) &= 0 & y=0 & \text{or } \lambda=-2 \\ L'_z &= 3z^2 - \lambda \cdot 3z^2 = 0 & 3z^2(1-\lambda) &= 0 & z=0 & \text{or } \lambda=1 \end{aligned}$$

CSC: $\lambda \geq 0$

C: $x^3 + y^3 + z^3 = 8$ if $\lambda > 0$

Candidate pts:

i) $x=y=z=0$, $x^3+y^3+z^3 < 0 < 8 \Rightarrow \lambda=0 \Rightarrow \boxed{(0,0,0;0)}$
 $f=0$

ii) $x=y=0, z \neq 0$ and $\lambda=1$:
 $x^3+y^3+z^3 = z^3 = 8 \Rightarrow z=2 \Rightarrow \boxed{(0,0,2;1)}$
 $f=8$

iii) $y=z=0, x \neq 0$ and $\lambda=4$:
 $x^3+y^3+z^3 = x^3 = 8 \Rightarrow x=2 \Rightarrow \boxed{(2,0,0;4)}$
 $f=32$

iv) $\lambda=4$ and $\lambda=1$: impossible

Best candidate point: $(x,y,z;\lambda) = \underline{\underline{(2,0,0;4)}} \quad f=32$

b) There is no max since $f \rightarrow \infty$ when $x=0, y \rightarrow -\infty$ and $z=0$. For example,

$$f(0,-3,0) = 81 > 32$$

Moreover $(0,y,0)$ is admissible when $y \leq 2$.

3. $\max f(x,y,z) = 2x^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \leq 16$

a) $x^4 + y^4 + z^4 \leq 16$ defines a bounded set,
 Since $-2 \leq x, y, z \leq 2$.
 \Downarrow EVT

there is a max.

b) NDCQ:
 a) $x^4 + y^4 + z^4 < 16$: no condition
 b) $x^4 + y^4 + z^4 = 16$:
 $\text{rk } f = \text{rk} (4x^3 \ 4y^3 \ 4z^3) = 1$
 NDCQ fails if $x=y=z=0$, not adm.

NDCQ holds
 for all adm.
 pts.

c) From a), b) it follows that there is a max at a regular candidate pt, with FOC + CSC satisfied.
 $h = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4)$

FOC:

$$\begin{aligned} h'_x &= 4x - \lambda \cdot 4x^3 = 0 & 4x(1 - \lambda x^2) &= 0 \\ h'_y &= -8y - \lambda \cdot 4y^3 = 0 & 4y(-2 - \lambda y^2) &= 0 \\ h'_z &= -4z - \lambda \cdot 4z^3 = 0 & 4z(-1 - \lambda z^2) &= 0 \end{aligned}$$

C:
 +

a) $x^4 + y^4 + z^4 < 16$: $\lambda = 0 \Rightarrow x = y = z = 0$

$(0, 0, 0; 0)$
 $f = 0$

CSC:

b) $x^4 + y^4 + z^4 = 16$: $\lambda \geq 0$

$\lambda = 0$ or $\lambda = 1/x^2$

$y = 0$ or $\lambda = -2/y^2 < 0 \Rightarrow \underline{y = 0}$

$z = 0$ or $\lambda = -1/z^2 < 0 \Rightarrow \underline{z = 0}$

$x = 0 \Rightarrow (x, y, z) = (0, 0, 0)$ not adm. impossible.

$\lambda = 1/x^2 \Rightarrow x^2 = 1/\lambda \Rightarrow x = \pm \sqrt{1/\lambda}$

$x^4 + y^4 + z^4 = (1/\lambda)^2 + 0 + 0 = 16 \Rightarrow \lambda^2 = 1/16 \Rightarrow \lambda = 1/4$

$\Rightarrow (\pm 2, 0, 0; 1/4)$
 $f = 8$

Best
 cand.
 pt $\Rightarrow \underline{f_{\max} = 8}$

$(x, y, z) = (\underline{\pm 2}, 0, 0)$

d) $\max f(x,y,z) = ax^2 - 4y^2 - 2z^2$ when $g(x,y,z) = x^4 + y^4 + z^4 - b = 0$

Note: 1) $a=2, b=16$: $f^*(2,16) = 8$
 $x^* = \pm 2, y^* = z^* = 0, z^* = 1/4$

2) the problem has a max which is regular cond. pt. as long as $a, b > 0$.

Envelope thm:

i) $a=2, b=20$: $f^*(2,20) = 8 + 4 \cdot \frac{df^*}{db} = 8 + 4 \cdot \frac{1}{4} = \underline{\underline{9}}$

ii) $a=1, b=16$: $f^*(1,16) = 8 + (-1) \cdot (\pm 2)^2 = \underline{\underline{4}}$

iii) $a=1, b=20$: $f^*(1,20) = 8 + (-1) \cdot (\pm 2)^2 + 4 \cdot 1/4 = \underline{\underline{5}}$

We use that:

$$h = ax^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - b)$$

$$\frac{\partial h}{\partial a} = x^2 \Rightarrow \frac{df^*}{da} = x^*(a,b)^2$$

$$\frac{\partial h}{\partial b} = \lambda \Rightarrow \frac{df^*}{db} = \lambda^*(a,b)$$

(Exact values: $f^*(2,20) = 4\sqrt{5} \approx 8.94$)
 $f^*(1,16) = 4$
 $f^*(1,20) = 2\sqrt{5} \approx 4.47$