Solutions: Key problems for Lecture 6

1. a) $A = \begin{pmatrix} 5 & 3 & 8 \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}$

$D_1 = 5$
$D_2 = 10 - 9 = 1$
$D_3 = 13 \cdot 1 - 5 \cdot (25-24) + 8 \cdot (15-16) = 13 - 5 - 8 = \underline{0}$

Rec: $\operatorname{rk} A = 2$, $D_1, D_2 > 0$ $\Rightarrow$ pos. semidefin

b) $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$

$D_1 = 1$
$D_2 = 1$
$D_3 = D_4 = 0$ since $\operatorname{rk} A = 2$

Rec: positive semidefin.

c) $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

$D_1 = 0$
$D_2 = -1 \Rightarrow$ indefinite

d) $A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$

$D_1 = 1 > 0$
$D_2 = 1 - 1/4 = 3/4 > 0$
$D_3 = 1 \cdot \frac{3}{4} - 1/2 \cdot 1/2 = 2/4 > 0$
$D_4 = 1 \cdot (\frac{2}{4}) - 1/2 \cdot 1/2 \cdot (1 - 1/4)$
$= 2/4 - 1/4 \cdot 3/4 = \frac{8-3}{16} = \frac{5}{16} > 0$

$\Rightarrow$ pos. defin.

2. $D_1 = a \le 0 \Rightarrow a \le 0$

$D_2 = a^2 \ge 0$ ok.

$D_3 = a \cdot (a^2 - 1) \le 0 \Rightarrow a = 0$, ok.
or
$a < 0$, $a^2 - 1 \ge 0 \Rightarrow a^2 \ge 1 \Rightarrow a \le -1$

$D_4 = a \cdot a(a^2-1) + 1 \cdot (-1)(a^2-1)$
$= (a^2-1) \cdot (a^2-1) \ge 0$ ok.

This means:
A neg. semidefin
$\Downarrow$
$a = 0$ or $a \le -1$

(other implication)

Check: $a = 0$ $\Delta_2 = M_{23,23} = -1 < 0 \Rightarrow A$ indefinite

$a \le -1$: $\Delta_1 = a, a, a, a \le 0$ ✓
$\Delta_2 = a^2, a^2, a^2-1, a^2-1, a^2, a^2 \ge 0$ ✓
$\Delta_3 = a(a^2-1), a(a^2-1), a(a^2-1), a(a^2-1) \le 0$ ✓
$\Delta_4 = (a^2-1)^2 \ge 0$ ✓

Conclusion:

A negative semidefn. $\Longleftrightarrow$ $\underline{\underline{a \le -1}}$

3. a) $f = xy + xz - yz$

$$\left.\begin{aligned} f'_x &= y + z = 0 \\ f'_y &= x - z = 0 \\ f'_z &= x - y = 0 \end{aligned}\right\} \Rightarrow H(t) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\Downarrow$

$x = z,\ y = -z$

$x - y = z + z = 0$

$z = 0$

$x = y = 0$

Stat. pt: $\underline{\underline{(0,0,0)}}$

$$\left.\begin{aligned} D_1 &= 0 \\ D_2 &= -1 \end{aligned}\right\} \begin{array}{l} H(t) \\ \text{indefinite} \end{array}$$

$\Downarrow$

$\underline{(0,0,0) \text{ saddle pt.}}$

$\underline{\underline{\text{no global max/min}}}$

b) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

$$\left.\begin{aligned} f'_x &= 2x + y = 0 \\ f'_y &= 2y + x + z = 0 \\ f'_z &= 2z + y + w = 0 \\ f'_w &= 2w + z = 0 \end{aligned}\right\} \Rightarrow H(t) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$y = -2x$

$z = -x - 2(-2x) = 3x$

$w = 2x - 2(3x) = -4x$

$2(-4x) + 3x = 0$

$-5x = 0 \Rightarrow x = 0$

$x = y = z = w = 0$

Stat. pt: $(x,y,z,w) = \underline{\underline{(0,0,0,0)}}$

$$\left.\begin{aligned} D_1 &= 2 \\ D_2 &= 3 \\ D_3 &= 2\cdot 3 - 1\cdot 2 = 4 \\ D_4 &= 2\cdot 4 - 1\cdot 1\cdot 3 = 5 \end{aligned}\right\} \begin{array}{l} H(t) \text{ pos.} \\ \text{defn. for} \\ \text{all } (x,y,z,w) \end{array}$$

$\Downarrow$

$\underline{(0,0,0,0) \text{ local min}}$ (second der. test)

f convex $\Rightarrow$ $\underline{(0,0,0,0) \text{ global min}}$

$f_{min} = f(0,0,0,0) = \underline{\underline{0}}$

(no global max)

c) $f = x^4 + y^4 + z^4 + z^2$

$$\left.\begin{aligned} f'_x &= 4x^3 = 0 \\ f'_y &= 4y^3 = 0 \\ f'_z &= 4z^3 + 2z = 0 \end{aligned}\right\}$$

$$H(f) = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 12y^2 & 0 \\ 0 & 0 & 12z^2+2 \end{pmatrix}$$

$x = 0$, $y = 0$

$2z(2z^2+1) = 0$

$z = 0$ or $2z^2 + 1 = 0$ no. sol.

$\Downarrow$

$x=0$,
$y=0$
$z=0$

Stat. pt.: $(x,y,z) = \underline{\underline{(0,0,0)}}$

$$H(f)(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

pos. semidefin.
no conclusion
(from second derivative test)

$H(f)$: $D_1 = 12x^2 \geq 0$

$D_2 = 144x^2y^2 \geq 0$

$D_3 = 144x^2y^2(12z^2+2) \geq 0$

$H(f)$ pos. semidefin. for all (x,y,z)

$\Downarrow$

f convex, $(0,0,0)$ global min

$\Downarrow$

$(0,0,0)$ local min

$f_{min} = f(0,0,0) = \underline{\underline{0}}$

(no global max)

---

d) $f = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

$$\left.\begin{aligned} f'_x &= -4x^3 - 4x + 6z = 0 \\ f'_y &= -6y = 0 \\ f'_z &= 6x - 12z = 0 \end{aligned}\right\}$$

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$y = 0$, $z = \frac{6x}{12} = \frac{x}{2}$

$-4x^3 - 4x + 6\cdot\left(\frac{x}{2}\right) = 0$

$-4x^3 - 4x + 3x = 0$

$-4x^3 - x = 0$

$-x(4x^2+1) = 0$

$x = 0$ or $4x^2 + 1 = 0$ impossible

$\Downarrow$

$y = 0$
$z = 0$

Stat. pts: $(x,y,z) = \underline{\underline{(0,0,0)}}$

$D_1 = -12x^2 - 4 < 0$ for all x,y,z

$D_2 = -6 \cdot D_1 > 0$ — " —

$D_3 = -6\cdot(144x^2 + 48 - 36) = -6(144x^2+12) < 0$ for all x,y,z

$\Downarrow$

$H(f)$ neg. defin. for all x,y,z

f concave, $(0,0,0)$ global max

$\Downarrow$

$(0,0,0)$ local max

$f_{max} = f(0,0,0) = \underline{16}$

(no global min)

4\. a) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

$H(f) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ pos. defin. for all x, y, z, w $\Rightarrow$ f **convex**

(from 3b.)

b) $f = e^{x-2y+z} = e^u$, $u = x - 2y + z$

$$\left.\begin{array}{l} f'_x = e^u \cdot 1 \\ f'_y = e^u \cdot (-2) \\ f'_z = e^u \cdot 1 \end{array}\right\} \Rightarrow H(f) = \begin{pmatrix} e^u & e^u \cdot (-2) & e^u \\ e^u(-2) & e^u \cdot 4 & e^u(-2) \\ e^u & e^u(-2) & e^u \end{pmatrix}$$

$$= e^u \cdot \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$D_1 = 1 \cdot e^u > 0$

$D_2 = 0 (e^u)^2$

RRC: $\text{rk}\, H(f) = 1$

$\Downarrow$

$H(f)$ pos. semidefin. for all (x, y, z)

$\Downarrow$

f **convex**

c) $f = x^4 + y^4 + z^4 + z^2$

f **convex** (from 3c)

d) $f = K - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

f **concave** (from 3d)

e) $f = \dfrac{xy + xz + yz}{xyz} = \dfrac{1}{z} + \dfrac{1}{y} + \dfrac{1}{x}$, $D_f: x, y, z > 0$

$$\left.\begin{array}{l} f'_x = -1/x^2 \\ f'_y = -1/y^2 \\ f'_z = -1/z^2 \end{array}\right\} \quad H(f) = \begin{pmatrix} 2/x^3 & 0 & 0 \\ 0 & 2/y^3 & 0 \\ 0 & 0 & 2/z^3 \end{pmatrix}$$

$\lambda_1 = 2/x^3 > 0$

$\lambda_2 = 2/y^3 > 0$

$\lambda_3 = 2/z^3 > 0$

for all x, y, z

$\Leftarrow$ f **convex**