Key Problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ:

a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 2.
Determine whether the matrix A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible:

a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ e) $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4.
Use eigenvalues and eigenvalues of A to determine the limit of A^m when $m \to \infty$, if the limit exists. What can you say about the limit of $A^m \cdot \mathbf{v}_0$, the equilibrium state of the Markov chain with transition matrix A?

a) $A = \begin{pmatrix} 0.30 & 0.15 \\ 0.70 & 0.85 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.86 & 0.42 \\ 0.14 & 0.58 \end{pmatrix}$ c) $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$

d) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ e) $A = \begin{pmatrix} 0.2 & 0.4 & 0 \\ 0.8 & 0.4 & 0.7 \\ 0 & 0.2 & 0.3 \end{pmatrix}$

Hint: To find eigenvalues and eigenvectors in c) and e), you can use a tool such as `https://www.wolframalpha.com/`. For example, `Eigenvalues of [[0.75,0.02,0.10],[0.2,0.9,0.2],[0.05,0.08,0.7]]` would give eigenvalues and eigenvectors in c).

Problems from the Workbook

Exam problems	Midterm exam GRA6035 10/2019 Question 1-5, 8
Workbook [W]	4.1 - 4.12 (full solutions in the workbook)
	4.13 - 4.15 (optional, harder problems)

Answers to Key Problems

Problem 1.

a) Eigenvalues $\lambda_1 = -4,\ \lambda_2 = 10$ and eigenvectors $E_{-4} = \text{span}(\mathbf{v}_1)$ and $E_{10} = \text{span}(\mathbf{v}_2)$, where $\mathbf{v}_1 = \begin{pmatrix}-1 & 1\end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix}1 & 1\end{pmatrix}^T$

b) Eigenvalues $\lambda_1 = \lambda_2 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix}1 & 1\end{pmatrix}^T$

c) No eigenvalues or eigenvectors

d) Eigenvalues $\lambda_1 = \lambda_2 = 5,\ \lambda_3 = 3$ and eigenvectors $E_5 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix}0 & 1 & 0\end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix}1 & 0 & 1\end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix}-1 & 0 & 1\end{pmatrix}^T$

e) Eigenvalues $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 4$ and eigenvectors $E_1 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix}-1 & 1 & 0\end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix}-1 & 0 & 1\end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix}1 & 1 & 1\end{pmatrix}^T$

f) Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$

Problem 2.

a) Yes, with $P = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$, $D = \begin{pmatrix}-4 & 0\\ 0 & 10\end{pmatrix}$

b) No

c) No

d) Yes, with $P = \begin{pmatrix}0 & 1 & -1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$, $D = \begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 3\end{pmatrix}$

e) Yes, with $P = \begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$, $D = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{pmatrix}$

f) No

Problem 3.
The eigenvalues of A are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = 2$.

Problem 4.
In all questions except d), the limit of $A^m\mathbf{v}$ is the vector $\mathbf{v}$ given below, and the limit of A^m is a matrix with the vector $\mathbf{v}$ in each column. In question d), A^m does not converge to a limit.

a) $\mathbf{v} = \begin{pmatrix}3/17\\ 14/17\end{pmatrix}$.

b) $\mathbf{v} = \begin{pmatrix}3/4\\ 1/4\end{pmatrix}$

c) $\mathbf{v} = \begin{pmatrix}2/15\\ 10/15\\ 3/15\end{pmatrix}$

d) None

e) $\mathbf{v} = \begin{pmatrix}7/25\\ 14/25\\ 4/25\end{pmatrix}$