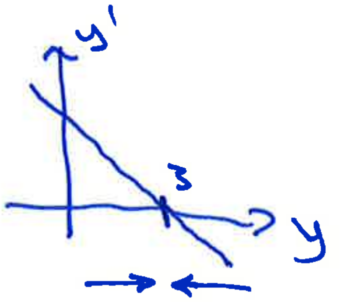
Solutions: Key Problems Lecture 11

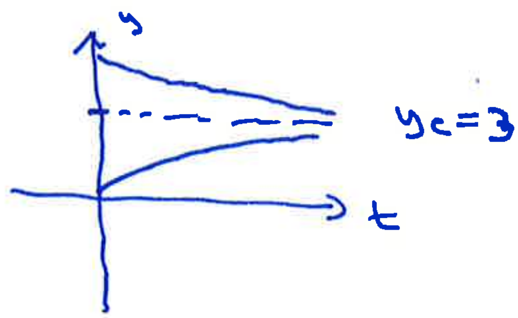
1. a) $y' = 6 - 2y$

Eq. state: $6 - 2y = 0$

$y = 3 \Rightarrow y_e = 3$



phase diagram



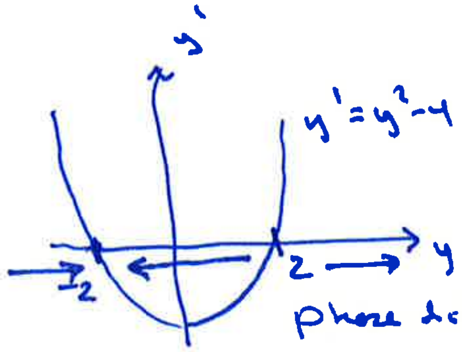
$y_e = 3$ is globally asymptotically stable

b) $y' = y^2 - 4$

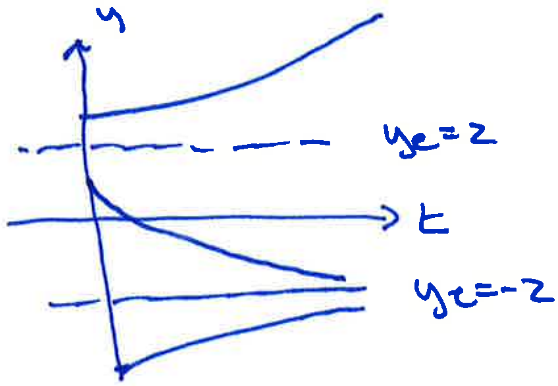
Eq. state: $y^2 - 4 = 0$

$y = \pm 2$

Eq. states: $y_e = -2$ and $y_e = 2$



phase diagram



$y_e = 2$ is unstable

$y_e = -2$ is stable, but not globally asymptotically stable

Alternative method: Stability thm.

$F(y) = y^2 - 4$

$F'(y) = 2y$

$F'(2) = 4 > 0 \Rightarrow y_e = 2$ unstable

$F'(-2) = -4 < 0 \qquad y_e = -2$ stable

note: cannot tell if $y_e = -2$ is globally asymptotically stable from $F'(-2)$.

c) $y' = 5y(1 - y/10)$

Eq. state:

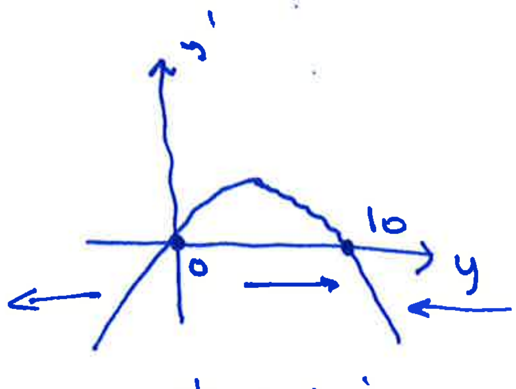
$5y(1 - y/10) = 0$

$y = 0$ or $1 - y/10 = 0$

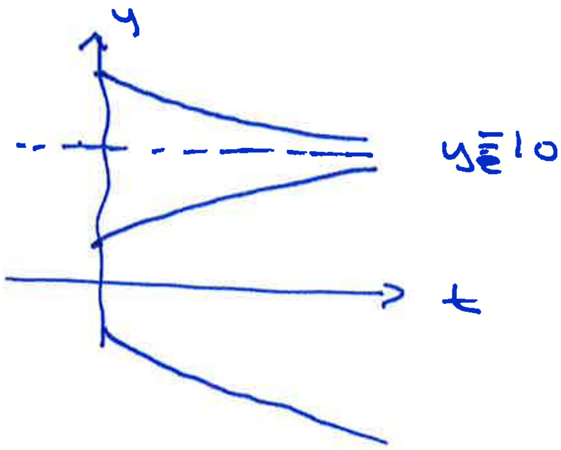
$y = 10$

Eq. states:

$y_e = 0$ and $y_e = 10$



phase diagram



$y_e = 0$ unstable

$y_e = 10$ stable, but not globally asymptotically stable

2. a) $y'' + 6y' - 16y = 16t - 22$

$$y = y_h + y_p = \underline{\underline{c_1 e^{2t} + c_2 e^{-8t} - t + 1}}$$

$\underline{y_h}$: $y'' + 6y' - 16y = 0$

$r^2 + 6r - 16 = 0$

$r = 2, \; r = -8 \implies y_h = \underline{c_1 e^{2t} + c_2 e^{-8t}}$

$\underline{y_p}$: $y'' + 6y' - 16y = 16t - 22$ ← $f(t) = 16t - 22$

$f'(t) = 16$

$f''(t) = 0$

$0 + 6 \cdot A - 16(At + B) = 16t - 22$ ← $\underline{\text{Guess}}$:

$y = \underline{At + B}$

$y' = A$

$y'' = 0$

$(-16A)t + (6A - 16B) = 16t - 22$

coeff. of t

const.

$\underline{\text{Comparison of coeff's:}}$

$-16A = 16 \implies A = \underline{-1}$

$6A - 16B = -22$

$-6 - 16B = -22$

$-16B = -16$

$\underline{B = 1}$

$$y_p = At + B = \underline{-t + 1}$$

b) $y'' + 6y' + 9y = 4e^{-t}$

$y = y_h + y_p = \underline{\underline{C_1 e^{-3t} + C_2 t e^{-3t} + e^{-t}}}$

$\underline{y_h}$: $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0$

$r = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 9}}{2}$

$= -3 \pm 0$

$r_1 = r_2 = \underline{-3}$ (double root) $\Rightarrow y_h = \underline{C_1 e^{-3t} + C_2 t e^{-3t}}$

y_p: $y'' + 6y' + 9y = 4e^{-t}$

$f = 4e^{-t}$
$f' = -4e^{-t}$
$f'' = 4e^{-t}$

$y = Ae^{-t}$ ← $\underline{\text{guess}}$
$y' = -Ae^{-t}$
$y'' = Ae^{-t}$

$(Ae^{-t}) + 6(-Ae^{-t})$
$+ 9(Ae^{-t}) = 4e^{-t}$

$(A - 6A + 9A)e^{-t} = 4e^{-t}$

$(4A)e^{-t} = 4e^{-t}$

$\underline{\text{Comparison of coeff's:}}$

$4A = 4$

$A = 1$

$y_p = Ae^{-t} = \underline{e^{-t}}$

c) $y'' - 3y' + 2y = 3e^{2t}$

$$y = y_h + y_p = \underline{C_1 e^{t} + C_2 e^{2t} + 3te^{2t}}$$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 1,\ r = 2 \quad \Longrightarrow \quad \underline{y_h = C_1 e^{t} + C_2 e^{2t}}$

y_p: $y'' - 3y' + 2y = 3e^{2t}$

$f = 3e^{2t}$ $\qquad$ $y = Ae^{2t}$ ← guess

$f' = 6e^{2t}$ $\qquad$ $y' = 2Ae^{2t}$

$f'' = 12e^{2t}$ $\qquad$ $y'' = 4Ae^{2t}$

$$(4Ae^{2t}) - 3(2Ae^{2t}) + 2(Ae^{2t}) = 3e^{2t}$$

$$(4A - 6A + 2A)e^{2t} = 3e^{2t}$$

$$0A e^{2t} = 3e^{2t}$$ ← no solution

try $y = t \cdot Ae^{2t} = \underline{Ate^{2t}}$

$y' = Ae^{2t} + At \cdot e^{2t} \cdot 2$

$= \underline{(A + 2At)e^{2t}}$

$y'' = 2Ae^{2t} + (A + 2At)e^{2t} \cdot 2$

$= \underline{(4A + 4At)e^{2t}}$

$$(4A + 4At)e^{2t} - 3(A + 2At)e^{2t} + 2 \cdot (At)e^{2t} = 3e^{2t}$$

$$(4A - 3A + \cancel{4At - 6At + 2At})e^{2t} = 3e^{2t}$$

$$Ae^{2t} = 3e^{2t}$$

$$\underline{A = 3}$$

$$y_p = Ate^{2t} = \underline{3te^{2t}}$$

d) $y'' - y = t^2$

$y = y_h + y_p = \underline{\underline{c_1 e^t + c_2 e^{-t} - t^2 - 2}}$

$\underline{y_h}$: $y'' - y = 0$

$r^2 - 1 = 0$

$r = \pm 1 \quad \Rightarrow \quad y_h = \underline{c_1 e^t + c_2 e^{-t}}$

$\underline{y_p}$: $y'' - y = t^2$ $\qquad f = t^2 \qquad$ $\underline{\text{Guess:}}$ $y = At^2 + Bt + C$

$f' = 2t \qquad y' = 2At + B$

$f'' = 2 \qquad y'' = 2A$

$2A - (At^2 + Bt + C) = t^2$

$(-A)t^2 + (-B)t + (2A - C) = t^2$

$\underline{\text{Comparison of coeff's:}}$ $(t^2 = 1 \cdot t^2 + 0 \cdot t + 0 \cdot 1)$

$-A = 1 \qquad A = \underline{-1}$

$-B = 0 \qquad B = \underline{0}$

$2A - C = 0 \qquad C = 2A = \underline{-2}$

$y_p = At^2 + Bt + C = \underline{-t^2 - 2}$