# Solutions: Key problems Lecture 10

**1.** a) $y' = 3t^2 + 2$

$y = \int 3t^2 + 2\,dt = \underline{t^3 + 2t + C}$

b) $ty' = 1$

$y' = 1/t$

$y = \int 1/t\,dt = \underline{\ln|t| + C}$

c) $y' = t\sqrt{t^2+1}$

$y = \int t\sqrt{t^2+1}\,dt = \int \cancel{t}\sqrt{u}\cdot\frac{du}{2\cancel{t}} = \int \frac{1}{2}u^{1/2}\,du$

$u = t^2+1$
$du = 2t\,dt$

$u\cdot\sqrt{u}$

$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C = \frac{1}{3}u^{3/2} + C = \underline{\underline{\frac{1}{3}(t^2+1)\sqrt{t^2+1} + C}}$

**2.** a) $y' = 5y$

$\frac{1}{y}y' = 5$

$\int \underbrace{\frac{1}{y}y'\,dt}_{dy} = \int 5\,dt$

$\int \frac{1}{y}\,dy = \int 5\,dt$

$\ln|y| = 5t + C$

have combined $C_1, C_2$ in the two integrals

$|y| = e^{5t+C} = e^C\cdot e^{5t}$

$y = \pm e^C e^{5t} = Ke^{5t}$

$y = \underline{\underline{Ke^{5t}}}$

b) $y' = y^2 t$

$\frac{1}{y^2}y' = t$

$\int \frac{1}{y^2}y'\,dt = \int t\,dt$

$\int y^{-2}\,dy = \int t\,dt$

$-y^{-1} = \frac{1}{2}t^2 + C$

$-\frac{1}{y} = \frac{1}{2}t^2 + C$

$\frac{1}{y} = -\frac{1}{2}t^2 - C$

$y = \frac{1}{-\frac{1}{2}t^2 - C}\frac{(-2)}{(-2)}$

$= \underline{\underline{\frac{-2}{t^2 + 2C}}}$

c) $y' = 5y(1 - y/10) = 5y \cdot \frac{1}{10} \cdot (10 - y)$

$$\frac{1}{y(10-y)} y' = 5 \cdot \frac{1}{10} = \frac{1}{2}$$

$$\int \frac{1}{y(10-y)} y' \, dt = \int \frac{1}{2} \, dt$$

$$\int \frac{1}{y(10-y)} \, dy = \frac{1}{2} t + C$$

$(10-y)' = -1$

$$\frac{1}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad \Big| \, y(10-y)$$

$$1 = A \cdot (10-y) + B \cdot y$$
$$= 10A - Ay + By$$
$$1 = 10A + (B-A)y$$

$\Downarrow$ compare coeff.

$B - A = 0$ (no $y$-term on LHS)

$10A = 1$ (const. term)

$\Downarrow$

$\underline{A = \frac{1}{10}}$ $\quad B = A = \underline{\frac{1}{10}}$

$$\frac{1}{y(10-y)} = \frac{1/10}{y} + \frac{1/10}{10-y}$$

$$\int \frac{1/10}{y} + \frac{1/10}{10-y} \, dy = \frac{1}{2} t + C$$

$$\frac{1}{10} \ln|y| - \frac{1}{10} \ln|10-y| = \frac{1}{2} t + C$$

$$\ln|y| - \ln|10-y| = 10\left(\frac{1}{2} t + C\right)$$

$$\ln\left|\frac{y}{10-y}\right| = 5t + 10C$$

$$\left|\frac{y}{10-y}\right| = e^{5t + 10C}$$

$$\frac{y}{10-y} = \pm e^{10C} e^{5t} = K e^{5t}$$

$$y = K e^{5t} \cdot (10 - y)$$
$$y = 10 K e^{5t} - K e^{5t} \cdot y$$
$$y + K e^{5t} y = 10 K e^{5t}$$
$$\underline{\underline{y = \frac{10 K e^{5t}}{1 + K e^{5t}}}}$$

3. a) $y' + 3y = 6$

**Alt A:** integrating factor $\quad y' + a(t)y = b(t)$

$a(t) = 3 \Rightarrow \int a(t)dt = 3t + C \Rightarrow u = \underline{\underline{e^{3t}}}$

$(y' + 3y)e^{3t} = 6e^{3t}$

follows from choice of $u$ (theory)

$(y \cdot e^{3t})' = 6e^{3t}$

$$y e^{3t} = \int 6e^{3t} dt = 6 \cdot \frac{1}{3}e^{3t} + C = 2e^{3t} + C$$

$$y = \frac{2e^{3t} + C}{e^{3t}} = 2 + \frac{C}{e^{3t}} = \underline{\underline{2 + C \cdot e^{-3t}}}$$

**Alt B:** superposition principle

$y = y_h + y_p = \underline{\underline{C \cdot e^{-3t} + 2}}$

$\underline{y_h}$: $y' + 3y = 0$

$r + 3 = 0$

$r = -3 \quad \Rightarrow \quad y_h = C \cdot e^{-3t}$

$\underline{y_p}$: $y' + 3y = 6 \quad$ Try: $y = A$ (const.)

$y' = 0$

$\Downarrow$

$y' + 3y = 6$

$0 + 3A = 6$

$\underline{A = 2} \Rightarrow y_p = 2$

b) $y' - 2ty = 4t$ $\qquad u = e^{\int -2t\,dt} = e^{-t^2+C} = \underline{e^{-t^2}}$

$(ye^{-t^2})' = 4te^{-t^2}$

$ye^{-t^2} = \int 4te^{-t^2}\,dt = \int \cancel{4t}e^{u}\frac{du}{-2\cancel{t}} = -2\int e^{u}\,du$

$$\boxed{\begin{matrix} u = -t^2 \\ du = -2t\,dt \end{matrix}}$$

$ye^{-t^2} = -2e^{u} + C = -2e^{-t^2} + C$

$y = \dfrac{-2e^{-t^2} + C}{e^{-t^2}} = -2 + \dfrac{C}{e^{-t^2}} = \underline{\underline{-2 + Ce^{t^2}}}$

c) $y' + 2y = e^{t}$

__Alt A:__ $u = e^{\int 2\,dt} = e^{2t}$

$(ye^{2t})' = e^{t}\cdot e^{2t} = e^{3t}$

$ye^{2t} = \int e^{3t}\,dt = \frac{1}{3}e^{3t} + C$

$y = \frac{1}{3}\frac{e^{3t}}{e^{2t}} + \frac{C}{e^{2t}} = \underline{\underline{\frac{1}{3}e^{t} + Ce^{-2t}}}$

__Alt B:__

$y = y_h + y_p = \underline{\underline{Ce^{-2t} + \frac{1}{3}e^{t}}}$

__$y_h$:__ $y' + 2y = 0$

$r + 2 = 0$

$r = -2$ $\Rightarrow y_h = Ce^{-2t}$

__$y_p$:__ $y' + 2y = e^{t}$

__Try:__ $y = Ae^{t}$ (A const.)

$y' = Ae^{t}$

$y' + 2y = e^{t} \Rightarrow Ae^{t} + 2(Ae^{t}) = e^{t}$

$3Ae^{t} = e^{t}$

$A = 1/3$ $\qquad 3A = 1 \Rightarrow A = 1/3$

$y_p = \frac{1}{3}e^{t}$

4. a) $3t^2-2t+2y\cdot y'=0$

<u>Exact?</u> $p=3t^2-2t \quad =h'_t$

$q=2y \quad =h'_y$

$\Downarrow$

I: $h=\int 3t^2-2t\,dt=t^3-t^2+g(y)$

II: $h'_y=0+g'(y)=2y \Rightarrow g(y)=y^2+C=y^2 \leftarrow$ may choose $C=0$

$\Downarrow$

$h=t^3-t^2+y^2$ satisfies I·II, eqn. is <u>exact</u>

<u>Solution:</u> $t^3-t^2+y^2=C \leftarrow h(t,y)=C$

$y^2=C+t^2-t^3$

$$y=\pm\sqrt{C+t^2-t^3}$$

b) $2y-3t^2+2(y+t)y'=0$

$p=2y-3t^2 \quad =h'_t \quad \Rightarrow h=2yt-t^3+g(y)$

$q=2y+2t \quad =h'_y \qquad h'_y=2t-0+g'(y)=2y+2t$

$g'(y)=2y$

$g(y)=y^2+C=y^2$

$h=2yt-t^3+y^2$ <u>exact</u>

$2yt-t^3+y^2=C$

$y^2+2t\cdot y+(-t^3-C)=0$

$$y=\frac{-2t\pm\sqrt{4t^2-4\cdot(-t^3-C)}}{2}$$

$$=-t\pm\sqrt{t^2+t^3+C}$$

c) $\frac{y(1-2\ln t)}{t^3} + \frac{\ln t}{t^2} y' = 0$

$p = \frac{y(1-2\ln t)}{t^3} = h'_t$

$q = \frac{\ln t}{t^2} = h'_y \Rightarrow h = \frac{\ln t}{t^2} \cdot y + g(t)$

Start with second eqn:

$h'_t = \frac{\frac{1}{t} \cdot t^2 - \ln t \cdot 2t}{t^4} y + g'(t)$

$= \frac{t - 2t\ln t}{t^4} y + g'(t)$

$p = y \frac{(1-2\ln t)}{t^3} \leftarrow$ should equal p $= \frac{1-2\ln t}{t^3} y + g'(t)$

ok if $g'(t) = 0$

choose $\underline{g(t) = 0}$

$h = \frac{\ln t}{t^2} \cdot y = c$

$$y = \frac{ct^2}{\ln t}$$