

Solutions: Problem Set 1

1.

$$a) \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 2 & -1 & 3 & 3 \\ 3 & 2 & 5 & 12 \end{array} \right) \begin{array}{l} \downarrow -2 \\ \leftarrow -3 \end{array} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 0 & \textcircled{-7} & -5 & -19 \\ 0 & -7 & -7 & -21 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 0 & \textcircled{-7} & -5 & -19 \\ 0 & 0 & \textcircled{-2} & -2 \end{array} \right)$$

echelon form

$$\begin{aligned} x &= 11 - 3 \cdot 2 - 4 \cdot 1 = \underline{1} \\ -7y &= -19 + 5 \cdot 1 = -14 \Rightarrow y = \underline{2} \\ -2z &= -2 \Rightarrow z = \underline{1} \end{aligned}$$

$$(x, y, z) = \underline{\underline{(1, 2, 1)}}$$

$$b) \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 2 & -1 & 3 & 3 \\ 3 & 2 & 7 & 12 \end{array} \right) \begin{array}{l} \downarrow -2 \\ \leftarrow -3 \end{array} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 0 & \textcircled{-7} & -5 & -19 \\ 0 & -7 & -5 & -21 \end{array} \right) \downarrow -1$$

$$\rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 3 & 4 & 11 \\ 0 & \textcircled{-7} & -5 & -19 \\ 0 & 0 & 0 & \textcircled{-2} \end{array} \right)$$

echelon form

no solutions

$$c) \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 8 \\ 1 & 3 & 1 & 5 & 28 \\ 2 & 4 & 2 & 9 & 48 \end{array} \right) \begin{array}{l} \downarrow -1 \\ \leftarrow -2 \end{array} \rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 8 \\ 0 & \textcircled{2} & 0 & 4 & 20 \\ 0 & 2 & 0 & 7 & 32 \end{array} \right) \downarrow -1$$

$$\rightarrow \left(\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 1 & 8 \\ 0 & \textcircled{2} & 0 & 4 & 20 \\ 0 & 0 & 0 & \textcircled{3} & 12 \end{array} \right)$$

echelon form
(z free)

$$x = 8 - 2 - z - 4 = \underline{2 - z}$$

$$2y = 20 - 4 \cdot 4 = 4 \Rightarrow y = 2$$

$$3w = 12 \Rightarrow \underline{w = 4}$$

$$(x, y, z, w) = \underline{\underline{(2 - z, 2, z, 4)}}$$

$$\underline{2.} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 1 & 2 & 4 & 13 \\ 1 & 3 & 9 & 24 \end{array} \right) \begin{array}{l} \downarrow -1 \\ \leftarrow -1 \end{array} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 6 \\ 0 & \textcircled{1} & 2 & 7 \\ 0 & 2 & 7 & 18 \end{array} \right) \downarrow -2$$

$$\rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & 2 & 6 \\ 0 & \textcircled{1} & 2 & 7 \\ 0 & 0 & \textcircled{3} & 4 \end{array} \right)$$

one unique solution
when $a =$ blue coeff.
 $= 2$

$$\left(\begin{array}{ccc|c} \textcircled{1} & 1 & a & 6 \\ 1 & 2 & 4 & 13 \\ 1 & 3 & 9 & 24 \end{array} \right) \begin{array}{l} \downarrow -1 \\ \leftarrow -1 \end{array} \rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & a & 6 \\ 0 & \textcircled{1} & 4-a & 7 \\ 0 & 2 & 9-a & 18 \end{array} \right) \downarrow -2$$

$$\rightarrow \left(\begin{array}{ccc|c} \textcircled{1} & 1 & a & 6 \\ 0 & \textcircled{1} & 4-a & 7 \\ 0 & 0 & 1+a & 4 \end{array} \right)$$

$a \neq -1$: one unique solution
 $a = -1$: no solutions

$$\underline{3.} \quad \left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ 5 & 5 & -1 & 4 & 0 \\ 7 & 6 & 3 & 3 & 0 \end{array} \right] \xrightarrow[-7]{-5} \left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ 0 & 0 & -21 & 9 & 0 \\ 0 & -1 & -25 & 10 & 0 \end{array} \right] \updownarrow$$

$$\rightarrow \left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ 0 & \textcircled{-1} & -25 & 10 & 0 \\ 0 & 0 & \textcircled{-21} & 9 & 0 \end{array} \right]$$

When $t = \text{red coeff} = 5$,
the solutions form
a line in \mathbb{R}^4 (1-dim)
and we have

echelon form
one free variable = one degree of freedom
(infinitely many sol'n)

Note: it is not necessary to include
the last column for homogeneous
systems

$$\left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ t & 5 & -1 & 4 & 0 \\ 7 & 6 & 3 & 3 & 0 \end{array} \right] \xrightarrow[-7]{-t} \left[\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 0 \\ 0 & 5-t & -1-4t & 4+t & 0 \\ 0 & -1 & -25 & 10 & 0 \end{array} \right] \updownarrow$$

$$\left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ 0 & \textcircled{-1} & -25 & 10 & 0 \\ 0 & 5-t & -1-4t & 4+t & 0 \end{array} \right] \xrightarrow{t(5-t)} \left[\begin{array}{cccc|c} \textcircled{1} & 1 & 4 & -1 & 0 \\ 0 & \textcircled{-1} & -25 & 10 & 0 \\ 0 & 0 & 21t-126 & 54-9t & 0 \end{array} \right]$$

$$\left. \begin{array}{l} 21t-126=0 \\ \quad t=6 \\ \quad \parallel \\ 54-9t=0 \end{array} \right\}$$

$t=6$: two free var. (z, w)
 $t \neq 6$: one free var. (w)
