Plan:

1. Vectors and span
2. Linear subspaces and geometry
3. Linear independence
4. Dimension of linear subspaces and rank

Readings:

[MEJ] 10.1-10.3, (10.4-10.7), 11.1-11.4, (27.1+27.5)

Review:

- matrix multiplication and identity matrix I
- inverse matrix A^{-1}: $|A| \neq 0 \iff A^{-1}$ exists
- determinants and minors

$\underset{n\times n}{A}$: $\begin{cases} rk(A) = n & : \quad |A| \neq 0 \\ rk(A) < n & \quad\;\; |A| = 0 \end{cases}$

$\underset{m\times n}{A}$: $\begin{cases} rk(A) = r & : \text{ there is at least one } r\text{-minor} \neq 0 \\ rk(A) < r & : \text{ all } r\text{-minors are zero} \end{cases}$

r = size of a maximal minor

- Linear systems:

$$x+y+z+w = 4$$
$$2x+3y-z+2w = 7$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 2 & 3 & -1 & 2 & 7 \end{array}\right)$$

$M_{12,12} = 3-2 = 1 \neq 0$: x, y basic, z, w free

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4-z-w \\ 7+z-2w \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} * \\ * \end{pmatrix}$$

$M_{12,34} = 3 \neq 0$: z, w basic, x, y free

$M_{12,14} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$: cannot choose x, w basic, y, z free

1 Vectors and span

Defn: An m-vector (a vector in $\mathbb{R}^m$) is a column vector of the form

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix}$$

Operations: addition, subtraction, scalar multiplication

Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}$

$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underline{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}$, $-1 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \underline{\begin{pmatrix} -3 \\ 1 \end{pmatrix}}$

Defn: Let $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ be m-vectors

A linear combination of $\{\underline{v_1}, \ldots, \underline{v_n}\}$ is an expression of the form

an m-vector $\longrightarrow$ $c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2} + \ldots + c_n \cdot \underline{v_n} = x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + \ldots + x_n \cdot \underline{v_n}$

where $c_1, c_2, \ldots, c_n$ are numbers.

The span of $\{\underline{v_1}, \ldots, \underline{v_n}\}$ is the set of vectors

$$V = \text{span}(\underline{v_1}, \ldots, \underline{v_n}) = \{ c_1 \cdot \underline{v_1} + \ldots + c_n \cdot \underline{v_n} : c_1, c_2, \ldots, c_n \}$$

This is a subset of $\mathbb{R}^m$.

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$V = \text{span}(\underline{v}_1, \underline{v}_2) = \{ x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 \; : \; x_1, x_2 \text{ numbers} \}$

$$x_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

Q: Is $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ in $V = \text{span}(\underline{v}_1, \underline{v}_2)$?

$$\underbrace{\begin{pmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}}_{\text{vector eqn.}} \qquad \begin{aligned} x_1 + 3x_2 &= 7 \\ 2x_1 - x_2 &= 5 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 5 \end{array} \right) \begin{matrix} \\ \rfloor -2 \end{matrix} \to \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -9 \end{array} \right)$$

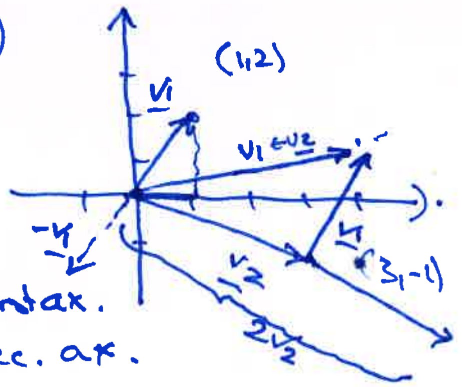
echelon form
one solution

Yes, $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ is in $V = \text{span}(\underline{v}_1, \underline{v}_2)$

Any vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is in $\text{span}(\underline{v}_1, \underline{v}_2) = V$ since the pivots would be the same.

Conclusion: $V = \text{span}(\underline{v}_1, \underline{v}_2) = \mathbb{R}^2$

② Geometric representations of vectors



Ex: 2-vectors $\underline{v_1} = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{v_2} = \begin{pmatrix}3\\-1\end{pmatrix}$

We can think of vectors are displacement

$\underline{v_1} = \begin{pmatrix}1\\2\end{pmatrix}$ means 1 unit along pos. first ax.
2 — 1 ——— sec. ax.

We can see { vector addition, scalar multiplication } geometrically.

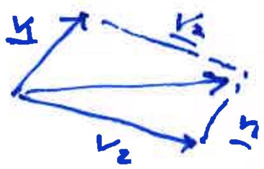
$$\underline{v_1} + \underline{v_2} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$$

$$-\underline{v_1} = -1 \cdot \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

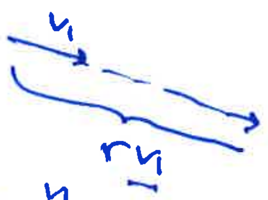
$$2\underline{v_2} = 2 \cdot \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix}$$

Displacement: $\underline{v} = \begin{pmatrix}v_1\\v_2\end{pmatrix}$ means
- move v_1 units along the positive first axis, then
- move v_2 units along the positive second axis

$\underline{v_1} + \underline{v_2}$:



$r \cdot \underline{v_1}$:

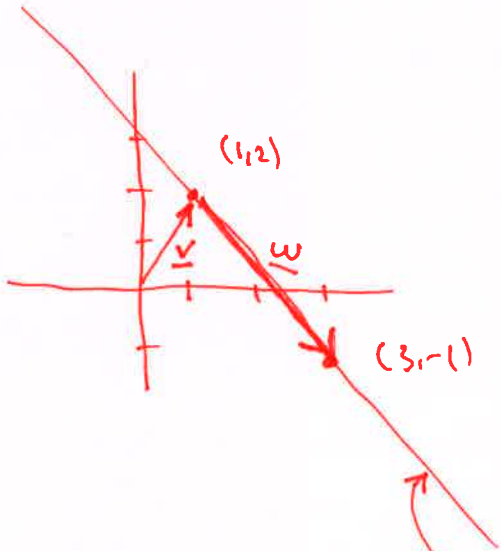


if $r > 0$

if $r < 0$

BREAK

Parametric description of a line through given points



Ex: Line through (1,2), (3,-1):

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ vector from (0,0) to (1,2)

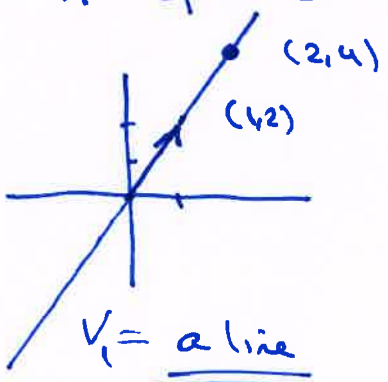
$\underline{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

vector from (1,2) to (3,-1)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \underline{v} + t \cdot \underline{w}$ ← vector from (0,0) to some point on the line

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1+2t \\ 2-3t \end{pmatrix}$

<u>Ex:</u> $V_1 = \text{span}(\underline{v_1})$ $\quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

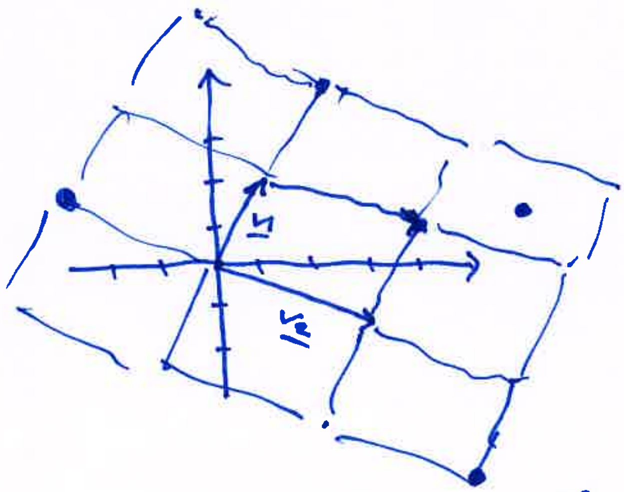


$\boxed{\begin{array}{ccc} (2,4) & \Longleftrightarrow & \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ \text{pt.} & & \text{vector} \end{array}}$

$(0,0) \to (2,4)$

$V_1 = $ <u>a line</u>

$V_2 = \text{span}(\underline{v_1}, \underline{v_2})$ $\quad \underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$



$\underline{v_1} + \underline{v_2}$

$2\underline{v_1} - \underline{v_2}$

$-\underline{v_2}$

$2\underline{v_2} - \underline{v_1}$

$V_2 = $ a <u>plane</u> $= \mathbb{R}^2$

<u>Defn.:</u> A <u>linear subspace</u> of $\mathbb{R}^m$ is a subset V of $\mathbb{R}^m$ such that

(linear property) $\longrightarrow$ i) If P, Q are in V with $P \neq Q$, then the line through P and Q is in V

ii) $(0, 0, \ldots, 0)$ is in V

Another way to say this:

a) If $\underline{v}, \underline{w}$ is in V, then $\underline{v} + \underline{w}$ is in V

b) If $\underline{v}$ in V, then $r \cdot \underline{v}$ is in V for all r.

Ex: i) $\boxed{V = \text{span}(\underline{v}_1, \dots, \underline{v}_n)}$ is a linear subspace

property i) + ii) of the second defn.

$\underline{v}_1 + \underline{v}_2$, $2\underline{v}_1 - 3\underline{v}_2$: $\underline{v}_1 + \underline{v}_2 + 2\underline{v}_1 - 3\underline{v}_2 = 3\underline{v}_1 - 2\underline{v}_2$

$7 \cdot (2\underline{v}_1 - 3\underline{v}_2) = 14\underline{v}_1 - 21\underline{v}_2$

ii) $\boxed{\text{The set of solutions of any homogeneous linear system}}$ is a linear subspace.

prop. i) and ii) of the first defn.

if P and Q are solutions, then the straight line thr. P and Q consists of solutions — linear property of any linear system.

$(0, 0, \dots, 0)$ is solution if the linear system is homogeneous

Ex:

$$x + y + z = 0$$
$$x + 2y + 4z = 0$$
$$2x + 3y + 5z = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right) \begin{matrix} \\ ]-1 \\ ]-2 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)]-1$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$$x + y + z = 0$$
$$y + 3z = 0$$

$y = -3z$

$x = -y - z = 2z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ -3z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \leftarrow \text{span}(\underline{w}), \quad \underline{w} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

linear subspace of $\mathbb{R}^3$

Solutions = a line

Ex: Linear system with two free parameters

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 2x_2 - x_3 + x_4 = 3$$
$$2x_1 + 3x_2 \qquad + 2x_4 = 7$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & -1 & 1 & 3 \\ 2 & 3 & 0 & 2 & 7 \end{array}\right) \begin{array}{l} -1 \\ -2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_2 - 2x_3 \qquad = -1$$

$$x_2 = -1 + 2x_3$$
$$x_1 = 4 - x_2 - x_3 - x_4$$
$$= 4 - (-1 + 2x_3) - x_3 - x_4$$
$$= 5 - 3x_3 - x_4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 - 3x_3 - x_4 \\ -1 + 2x_3 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_3 \\ 2x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix}$$

(x_3, x_4 free)

$$= \begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The homogeneous case:

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 2x_2 - x_3 + x_4 = 0$$
$$2x_1 + 3x_2 \qquad + 2x_4 = 0$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 1 & 0 \\ 2 & 3 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_3 - x_4 \\ 2x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

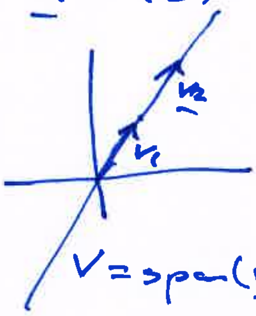
(a span of two vectors)

③ Linear independence

$\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$: m-vectors $\longrightarrow$ $V = \text{span}(\underline{v_1}, \dots, \underline{v_n})$ linear subspace

$\dim V = ?$

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



$V = \text{span}(\underline{v_1}, \underline{v_2})$ a line

$\underline{v_2} = 2\underline{v_1}$

$2\underline{v_1} - \underline{v_2} = 0$

$2\underline{v_1} = \underline{v_2}$

$\underline{v_1} = \frac{1}{2} \cdot \underline{v_2}$

$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2}$

$= x_1 \cdot \underline{v_1} + x_2 \cdot 2\underline{v_1}$

$= (x_1 + 2x_2) \cdot \underline{v_1}$

$$\boxed{\text{span}(\underline{v_1}, \underline{v_2}) = \text{span}(\underline{v_1})}$$

Defn: A linear dependency relation is an expression of the form

$$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + \dots + x_n \cdot \underline{v_n} = \underline{0}$$

where at least one $x_i \neq 0$.

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

Is one of the vectors a linear comb. of the others? $\longrightarrow$ Is there a linear dependency relation?

$\underline{v_1} = a \cdot \underline{v_2} + b \cdot \underline{v_3}$?

or

$\underline{v_2} = a \cdot \underline{v_1} + b\underline{v_3}$?

or

$\underline{v_3} = a \cdot \underline{v_1} + b\underline{v_2}$?

$x_1 \cdot \underline{v_1} + x_2 \cdot \underline{v_2} + x_3 \cdot \underline{v_3} = \underline{0}$

Defn: The vectors $\{\underline{v}_1, \dots, \underline{v}_m\}$ are linearly dependent if at least one of the vectors is a linear combination of the others. Otherwise, they are linearly independent.

linearly independent: none of the vectors are a linear combination of the others

$\boxed{\text{BREAK}}$

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix}1\\4\\-2\end{pmatrix}$

$$x_1 \cdot \underline{v}_1 + x_2 \cdot \underline{v}_2 + x_3 \underline{v}_3 = \underline{0}$$

$$x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\2\\0\end{pmatrix} + x_3 \cdot \begin{pmatrix}1\\4\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\boxed{\begin{matrix} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 4x_3 = 0 \\ x_1 \qquad - 2x_3 = 0 \end{matrix}}$$

$$\begin{array}{ccc|c} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \\ \end{array} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 0 & -2 & 0 \end{array}\right) \begin{matrix}]{-1} \\ \\ ]{-1} \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right)]_1 \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{matrix} x + y + z = 0 \\ y + 3z = 0 \end{matrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2z\\-3z\\z\end{pmatrix} = z \cdot \begin{pmatrix}2\\-3\\1\end{pmatrix}$$

$y = -3z$
$x = +3z - z = 2z$

$z = 1$: $\left.\begin{matrix} x_1 = 2 \\ x_2 = -3 \\ x_3 = 1 \end{matrix}\right\}$ $2\underline{v}_1 - 3\underline{v}_2 + 1 \cdot \underline{v}_3 = \underline{0}$

$$\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2$$

<u>Conclusions:</u> $\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ are <u>linearly dependent</u>

$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_1, \underline{v}_2)$

$\{\underline{v}_1, \underline{v}_2\}$ are <u>linearly independent</u>

<u>Method:</u> Linear independence

i) $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \rightsquigarrow A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n)$

m-vectors

$m \times n$-matrix

(coefficient matrix of $x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots + x_n\underline{v}_n = \underline{0}$)

ii) <u>Gaussian process:</u> $A \to \ldots \to E$ echelon form

note the pivot positions of E.

<u>Interpretation:</u>

If there are pivots in all columns of A : $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are <u>linearly independent</u>

If there are pivots in some columns of A, but not all of them:

Pivots in: $\{i_1, i_2, \ldots, i_r\}$

i) $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are <u>linearly dependent</u>

ii) $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ are <u>linearly independent</u> and all other vectors are in <u>$\text{span}\{v_{i_1}, v_{i_2}, \ldots\}$</u>

The case $m=n$: $A = (\underline{v_1} | \dots | \underline{v_n})$
$n \times n$-matrix (square)

i) $|A| \neq 0$: $\{\underline{v_1}, \dots, \underline{v_n}\}$ linearly **independent**

ii) $|A| = 0$: $\{\underline{v_1}, \dots, \underline{v_n}\}$ linearly **dependent**

Ex: $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ $\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 0 & -2 \end{vmatrix} = 1 \cdot 2 - 2 \cdot 1 = \underline{0}$

$\{\underline{v_1}, \underline{v_2}, \underline{v_3}\}$ linearly **dependent**

4) Dimensions of linear subspaces and rank

(m-vector)

Summary: $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n} \rightsquigarrow A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$

i) **Rank:** $rk(A) =$ maximal number of linearly independent vectors among $\{\underline{v_1}, \dots, \underline{v_n}\}$.

ii) **Column space:** $Col(A) = span(\underline{v_1}, \underline{v_2}, \dots, \underline{v_n})$
linear subspace of $\mathbb{R}^m$

$$\boxed{\dim Col(A) = rk(A)}$$

iii) **Null space:** $Null(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$
solutions of hom. lin. sys, linear subspace of $\mathbb{R}^n$

$$\boxed{\dim Null(A) = n - rk(A)}$$

Defn: A base for a linear subspace V in $\mathbb{R}^n$ is a collection $\{\underline{v}_1, \ldots, \underline{v}_r\}$ of vectors in V such that:

i) $\text{span}(\underline{v}_1, \ldots, \underline{v}_r) = V$

ii) $\{\underline{v}_1, \ldots, \underline{v}_r\}$ are linearly independent

Then $\dim V = r$ (the number of vectors in a base).

Method: To find a base of $V = \text{span}(\underline{v}_1, \ldots, \underline{v}_n)$, we do:

i) $A = (\underline{v}_1 | \underline{v}_2 | \ldots | \underline{v}_n) \to \cdots \to E$ echelon form

ii) Keep vectors corresponding to pivots, delete all other vectors $\to$ Base $\mathcal{B}$ of V