Plan:

1. Matrices and matrix algebra
2. Determinants and minors
3. Rank and linear systems using minors

Reading:

[ME] 8.1–8.4, (8.5–8.6), 9.1–9.2 (9.3), 26.1–26.3 (26.4), 26.5

Review:

- linear systems and Gaussian elimination
- interpret pivot position / solutions
- rank / solutions
- geometry (linear property, dimension)

Comment on Lecture Notes in It's Learning:

- Chapter I is complete, with (corrected) solutions for all problems
- I will post more chapters as they are written, but there will not be time to write all chapters during this semester. All theory is of course covered in the textbook [ME] and all important theory is covered by handwritten notes from lectures

① Matrices and matrix algebra

Defn: An <u>m×n-matrix</u> is a rectangular array of numbers (m rows, n columns)

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{n} \Big\} m$$

capital letter = matrix

An <u>m-vector</u> (column vector) is a column of numbers (m rows)

$$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \Big\} m$$

boldface / underlined letter = vector

<u>Operations on matrices:</u>

- <u>addition/subtraction:</u> $A \pm B$ defined if A, B have the same size

<u>Ex:</u> $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 5 & 6 \end{pmatrix}$

- <u>scalar multiplication:</u> scalar = number

<u>Ex:</u> $2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \cdot 2$

also work for vectors

<u>notation:</u> $-A = -1 \cdot A$

0 = zero matrix (a matrix of zeros)

<u>Ex:</u> $-\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & +1 \end{pmatrix}$

– matrix multiplication

$\underset{m\times n}{A}\cdot\underset{n\times p}{B}$ defined if # cols(A) = # rows(B)

Ex:

$$1\cdot 1+2\cdot 0$$

$$\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 1\\ 0 & 4 & 1\end{pmatrix}=\begin{pmatrix}1 & 8 & 3\\ 3 & -4 & 2\end{pmatrix}$$

$$2\times 2 \quad = \quad 2\times 3$$

Notice: $AB\neq BA$

Ex: Linear system

$$A=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}$$

$$a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m$$

$$\underline{x}=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}\qquad \underline{b}=\begin{pmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{pmatrix}$$

Matrix form of the linear system: $\boxed{A\cdot\underline{x}=\underline{b}}$

$$\begin{pmatrix}a_{11}x_1+\dots+a_{1n}x_n\\ \vdots\\ a_{m1}x_1+\dots+a_{mn}x_n\end{pmatrix}=\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}\cdot\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix}$$

$$A\cdot\underline{x}$$

Notation: $I=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}$ identity matrix

$$\left.\begin{aligned}I\cdot A&=A\\ A\cdot I&=A\end{aligned}\right|\ \text{for any matrix } A$$

A matrix is square if $m=n$ (# rows = # cols)

Powers: $A^2=A\cdot A$ defined when

$A^3=A\cdot A\cdot A$ A is square

$\vdots$

- transpose:

$$\underset{m\times n}{A} \rightsquigarrow \underset{n\times m}{A^T}$$

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Defn: A is called symmetric if $A^T = A$
(this means that A is square, and $a_{ij} = a_{ji}$ for all $i \neq j$)

- inverse matrix:

Let A be an $n\times n$-matrix. An inverse of A is a matrix A^{-1} such that

$$A^{-1} \cdot A = I$$
$$A \cdot A^{-1} = I$$

Method:

$$(A \mid I) \underset{\text{Gaussian process}}{\rightarrow \dots \rightarrow} (E \mid B)$$

where E is a reduced echelon form of A

$E = I$: $A^{-1} = B$

$E \neq I$: A^{-1} does not exist

Linear systems and inverse matrices:

$A\underline{x} = \underline{b}$
(linear system in matrix form)

If A is square and A^{-1} exists, then $\underline{x} = A^{-1}\underline{b}$ (one solution)

$$\left(\begin{array}{c} A\underline{x} = \underline{b} \\ \cancel{A^{-1}A}\underline{x} = A^{-1}\underline{b} \end{array} \right)$$

Note: $(AB)^T = B^T \cdot A^T$ $AB \neq BA$

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

$\boxed{\text{BREAK}}$

2 Determinants

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$n \times n$ matrix

Defn: $\det(A) = |A|$

$$\det(A) = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \dots + a_{1n} \cdot C_{1n}$$

cofactor expansion along the first row

C_{ij}: cofactor in position (i,j)

n=2 case:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (n=2 \text{ case})}$$

Cofactor C_{ij}:

$$C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot \underbrace{M_{ij}}_{\text{minor}}$$

Sign:

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

minor: M_{ij} = determinant of the submatrix you get if you delete row i, col. j

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \underbrace{1}_{a_{11}} \cdot \underbrace{(+1) \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix}}_{C_{11}} + \underbrace{1}_{a_{12}} \cdot \underbrace{(-1) \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix}}_{C_{12}}$$

$$+ \underbrace{1}_{a_{13}} \cdot \underbrace{(+1) \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}_{C_{13}} = 1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot 6 - 1 \cdot 5 + 1 \cdot 1 = \underline{\underline{2}}$$

Properties:

i) The cofactor expansion along any row or column gives the same result, det(A).

ii) $|AB| = |A| \cdot |B|$

iii) $|A^T| = |A|$

Determinants and inverses:

A ($n \times n$)

- $\det(A) = 0$: A^{-1} does not exist.
- $\det(A) \neq 0$: A^{-1} exists and

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}^T$$

Computing determinants via Gaussian process:

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix}}_{=B} \begin{matrix} \\ \\ -2 \end{matrix} \rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}}_{=C}$ echelon form

$|B| = +1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = \underline{\underline{2}}$ $\qquad |C| = +1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 1 \cdot 1 \cdot 2$

Fact: $A \rightarrow B$ elementary row operation

i) switch two rows $\Rightarrow |B| = -|A|$

ii) multiply a row with $c \neq 0$ $\Rightarrow |B| = c \cdot |A|$

iii) add a mult. of one row to another row $\Rightarrow |B| = |A|$

Defn. A matrix is diagonal if $a_{ij} = 0$ for all $i \neq j$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

A matrix is upper triangular if $a_{ij} = 0$ for all $i > j$

$$E = \begin{pmatrix} 1 & 4 & 7 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

An echelon form is upper triangular

Fact: If A is upper triangular, then

$$|A| = \underbrace{a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}}_{\text{product of diagonal entries}}$$

Determinants and linear systems

$A\underline{x} = \underline{b}$ linear system ($n \times n$, square)

- $|A| \neq 0$: one solution $\underline{x} = A^{-1} \cdot \underline{b}$
  $A \to \ldots \to E$ $\quad |E| \neq 0$
- $|A| = 0$: infinitely many sol. or no solution
  $A \to \ldots \to E$ $\quad |E| = 0$

BREAK

3 Minors, rank and linear systems

Minors:

A $m \times n$ matrix

Defn: An r-minor is the determinant of an $r \times r$-submatrix of A.

Ex:

$$\begin{array}{c} A = \\ 3\times 4 \end{array} \begin{array}{c} \begin{pmatrix} 1 & 2 & 0 & 4 \\ 4 & -1 & 2 & 0 \\ 0 & 1 & 7 & 3 \end{pmatrix} \\ \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$r=3$: 3-minors

3-minors = maximal minors

$$M_{123,123} = \begin{vmatrix} 1 & 2 & 0 \\ 4 & -1 & 2 \\ 0 & 1 & 7 \end{vmatrix} = 1(-9) - 2(28) = \underline{-65}$$

$$M_{123,234} = \begin{vmatrix} 2 & 0 & 4 \\ -1 & 2 & 0 \\ 1 & 7 & 3 \end{vmatrix} = 2 \cdot 6 + 4 \cdot (-9) = \underline{-16}$$

$$M_{123,124} = \ldots$$

$$M_{123,134} = \ldots$$

Result:

A $m \times n$ matrix

$$\boxed{\operatorname{rk}(A) = \max \{ r : \text{there is a non-zero } r\text{-minor} \}}$$

(the rank is the order/size of a maximal non-zero minor)

Ex: $A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & 4 & 2 \\ 3 & 1 & 7 & 2 \end{pmatrix}$ $rk(A) = ?$

Maximal minors = 3-minors:

$M_{123,123} = 1\cdot(-11) - 2(2) + 3\cdot(5) = 0$

$M_{123,124} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 1\cdot(-4) - 2(-2) - 1\cdot 5 = -5 \neq 0$

$rk(A) = 3$ (and I also know the pivot positions)

Procedure: i) Find r such that r-minor = maximal minors

ii) An r-minor is $\neq 0 \Rightarrow rk(A) = r$

All r-minors are $0 \Rightarrow rk(A) < r$

Ex: Find $rk \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & a \\ 1 & 3 & 9 \end{pmatrix}$:

Max. minors = 3-minors: $|A| = 1\cdot(18-3a) - 1\cdot(9-a) + 1\cdot 1$

$= 10 - 2a$

$|A| = 0$: $a = 5$

$|A| \neq 0$: $a \neq 5$

$a = 5$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 9 \end{pmatrix}$

$\Big(\underset{\text{2-minor}}{M_{12,12}} = 2-1 = 1 \neq 0 \Rightarrow rk A = 2 \Big)$

Conclusion:

$a \neq 5$: $rk A = 3$

$a = 5$: $rk A = 2$

$$\boxed{\underline{\text{Result:}} \quad \underset{\substack{n\times n\\ \text{matrix}}}{A} \quad \begin{cases} |A| \neq 0: & \text{rk}\, A = n \\ |A| = 0: & \text{rk}\, A < n \end{cases}}$$

Ex: Linear system: How to use minors to find (the number of) solutions.

$$x + 2y + z = 0$$
$$2x - y + 3z = 0$$
$$x + 7y = 0$$

Homogeneous system with

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 7 & 0 \end{pmatrix}$$

i) there are solutions since the system is homogeneous

ii) Find rk(A):

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 7 & 0 \end{vmatrix} = 1 \cdot (6+1) - 7 \cdot (3-2) = 7 - 7 = 0$$

all 3-minors = max. minors are zero

$$\Rightarrow \operatorname{rk} A < 3$$

2-minors: $M_{12,12} = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5 \neq 0 \Rightarrow \operatorname{rk} A = 2$

Pivot positions:

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 2 \\ 2 & -1 \end{matrix}} & \begin{matrix} 1 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 7 \end{matrix} & 0 \end{pmatrix}$$

the 2-minor $\neq 0$ we found before

$$x + 2y = -z$$
$$2x - y = -3z$$

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -z \\ -3z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -z \\ -3z \end{pmatrix}$$

$$= \frac{1}{-5} \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -z \\ -3z \end{pmatrix}$$

$$\boxed{\begin{array}{l} \text{ignore row 3: outside 2-minor} \\ z \text{ is free: outside 2-minor} \end{array}}$$

(cont.) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -z \\ -3z \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} -z-6z \\ -2z+3z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -7z \\ z \end{pmatrix} = \begin{pmatrix} -7z/5 \\ z/5 \end{pmatrix}$$

Solution: Infinitely many solutions (one free variable)

since $n - rk(A) = 3 - 2 = 1$

Explicitly: z is free, and

$(x, y, z) = (-7z/5, z/5, z)$ z free

Note! $M_{12,12} = -5 \neq 0 \Rightarrow$ this 2×2 matrix is invertible and we can solve for x and y

Problem: Compute

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} :$$

Answer / Solution:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \begin{matrix} \leftarrow \\ \\ \\ \leftarrow \\ \end{matrix} \begin{matrix} -4 \\ \\ \\ -1 \\ \end{matrix} = \begin{vmatrix} 0 & -4 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -4 & 0 & -1 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -2 & 2 & 0 & 0 \end{vmatrix} = -(-2) \cdot \begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 6 & -2 & 0 \end{vmatrix} - 2 \begin{vmatrix} -4 & -1 & -1 \\ 2 & 1 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot 6 \cdot (1-(-1)) - 2 \cdot (-2) \cdot (4-(-2))$$

$$= 12 \cdot 2 + 4 \cdot 6 = 24 + 24 = \underline{\underline{48}}$$