Plan:

Reading:

[MES] 6.1, (6.2), 7.1–7.4, (7.5)

1. Linear systems and geometry
2. Gaussian elimination
3. Rank and the number of solutions

Intro to GRA 6035 Mathematics:

- make yourself familiar with the material on the web page (linked from It's L. / google GRA6035)
- I will use It's L. to give messages about lectures / problems / etc.

1. Linear systems and geometry

Ex:
$$\begin{cases} x+y+z+w = 7 \\ x-y+2z+3w = 13 \\ 2x+3y \quad -w = 5 \end{cases}$$

3×4 linear system (3 linear equations, 4 variables x, y, z, w)

Defn: A linear equation in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written

$$\boxed{a_1x_1 + a_2x_2 + \ldots + a_nx_n = b}$$

where $a_1, a_2, \ldots, a_n$ and b are given numbers.

A linear system in the variables $x_1, x_2, \ldots, x_n$ is a system of equations that can be written

An $m \times n$ linear system (general form) $\rightarrow$

$$\left.\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n}_{n} = b_m \end{cases}\right\} m$$

Defn: A <u>matrix</u> is a rectangular array of numbers

$m \times n$ matrix : m rows, n col's

Given an $m \times n$ linear system $\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$

we define its <u>coefficient matrix</u> A and <u>augmented matrix</u> $(A|\underline{b})$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad (A|\underline{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

Ex:

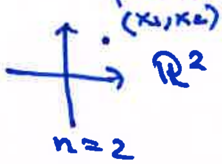
$$(*)\begin{cases} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{cases} \longrightarrow (A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$
$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$$

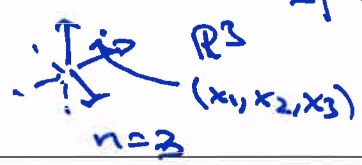
3×3 linear system

Defn: A <u>solution</u> of a linear system is a value for each of the variables such that all m equations are satisfied simultaneously.

Ex: $(x,y,z) = (1,1,1)$ is a solution of $(*)$

Geometry: Solutions of an $m \times n$ linear system consists of n-tuples $(x_1, x_2, \dots, x_n)$, which are points in n-dimensional space $\mathbb{R}^n$.





Let V be the set of all solutions of an $m \times n$ linear system. Then V is a geometric figure in n-dimensional space $\mathbb{R}^n$.

<u>$n=2$:</u>

(two var's)

$$ax + by = c \quad \begin{cases} \text{degenerate if } a=b=0 \\ \text{non-degenerate otherwise} \end{cases}$$

If the linear eq. is non-degenerate, its solutions is a <u>straight line</u>

$b \neq 0$: $\dfrac{by}{b} = \dfrac{c - ax}{b}$ $\qquad y = \dfrac{c}{b} - \dfrac{a}{b}x$

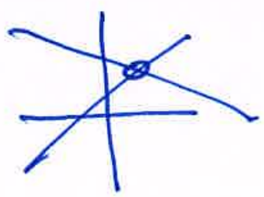
<u>line</u>

$b = 0, a \neq 0$: $ax = c$ $\qquad x = c/a$

<u>vertical line</u>

<u>Ex:</u> 2×2 linear systems $\qquad$ Solutions = intersection pts

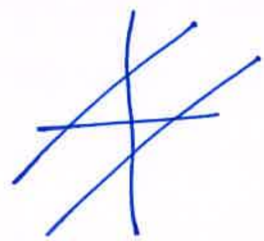
a)



one solution

V = a pt.

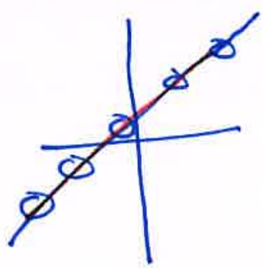
b)



no solutions

V = empty set

c)



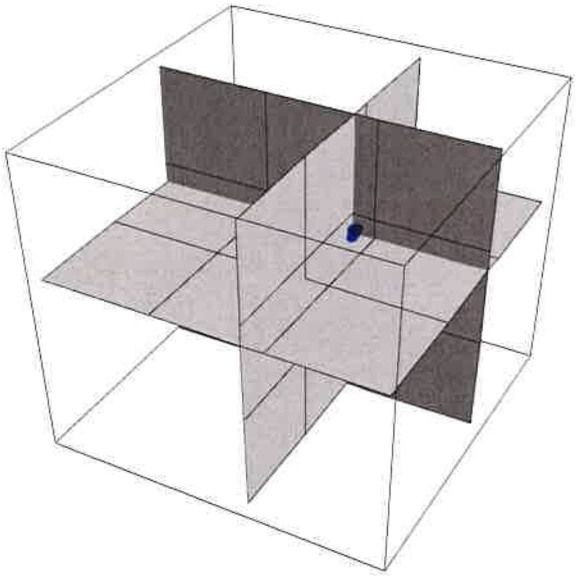
infinitely many solutions

V = a line

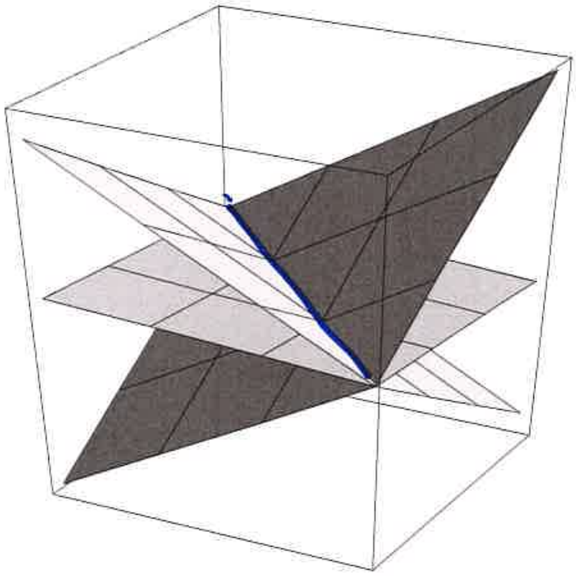
$\boxed{\text{BREAK}}$

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

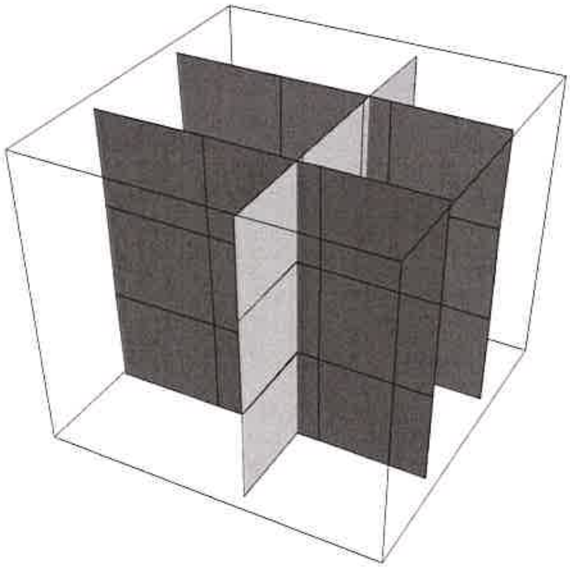
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*

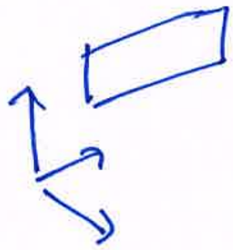


iii) There is not point in common to all three planes. *(no solution)*



n=3: (three var's) $ax + by + cz = d$ non-deg if $(a,b,c) \neq (0,0,0)$

a plane



General n: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

hyperplane

non-deg. if $(a_1, a_2, \dots, a_n) \neq (0, 0, \dots, 0)$

Theorem:

Any $m \times n$ linear system has either

i) no solution } inconsistent (no sol'n.)

ii) one solution }

iii) infinitely many solutions } consistent (at least one sol'n.)

Why? If P, Q are two different solutions of a linear syst., then all pts on the straight line thr. P and Q are also solutions.

$y = x + 2$

$y = x^2 + 2$

geometry: the set V of all solutions of a linear syst. is linear (not curved)

Polynomial equations of degree ≤ 1 are called linear because the set V of solutions is not curved (linear)

2 Gaussian elimination:

Method for solving linear systems
- general
- instructive
- fast

Steps:

1. write down the augmented matrix of the linear system
2. use elementary row operations to obtain an echelon form
3. write down the linear system of the echelon form
4. solve it by back substitution

Defn: Echelon form

Pivot = the first non-zero entry in a row

Echelon form: Matrix such that

i) zero rows are below non-zero rows

ii) all entries under a pivot are zero

Reduced echelon form: Matrix such that i), ii) + extra conditions hold:

iii) all pivots are 1

iv) all entries over a pivot are zero

Elementary row operations:

i) switch two rows

ii) multiply a row with $c \neq 0$

iii) add a multiple of one row to another row

- they do not change the solutions of the system
- can always find echelon form with these op.

Ex:

$$\begin{aligned} x+y+z&=3 \\ x+2y+4z&=7 \\ x+3y+9z&=13 \end{aligned} \quad \to \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} \cdot(-1) \\ + \\ + \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ \cdot(-2) \\ + \end{array}$$

$$\begin{aligned} x+y+z&=3 \\ y+3z&=4 \\ 2z&=2 \end{aligned} \qquad \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

$2z=2 \Rightarrow z=1$

$y+3z=4 \Rightarrow y = 4-3z = 1$

$x+y+z=3 \quad x = 3-y-z = 1$

$(x,y,z) = (1,1,1)$ One Solution

Back substitution: Solve each eqn. for the variable that corresponds to a pivot, backwards

Degenerate case: $0\cdot x_1 + 0\cdot x_2 + \dots + 0 \cdot x_n = b \qquad (0\ 0 \dots 0 \mid b)$

$b=0$: $0=0$ $\Rightarrow$ ignore this row

$b \neq 0$: $0=b$ $\Rightarrow$ the system has no solutions $\quad (0\ 0 \dots 0 \mid b)$

i) pivot in last column: no solutions

ii) all pivots are in the variable columns:

a variable is **basic** if there is a pivot in its column

a variable is **free** if there is no pivot in its column

Ex:

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & \boxed{2} & 2 \end{array}\right) \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x & y & z & w \end{array} \end{array}$$

x, y, w : basic

z : free

$$\begin{aligned} \underline{x} + 2y + 3z + 4w &= 5 \\ \underline{y} + 2z + 4w &= 0 \\ \underline{2w} &= 2 \end{aligned}$$

$$x = 5 - 2(-2z-4) \overset{-3z}{-} 4 = \underline{z+9}$$

$$y = -2z - 4w = \underline{-2z-4}$$

$$\underline{w = 1}$$

Solutions: $(x, y, z, w) = \underline{(z+9, -2z-4, z, 1)}$

where z is free

<u>infinitely many solutions</u>

<u>In this case:</u>

All variables are basic: <u>one solution</u>

At least one free variable: <u>infinitely many solutions</u>

BREAK

Extra worked example:

$$\begin{cases} x + y + z + w = 5 \\ 2x - y + 3z - w = 2 \\ x + 4y \qquad + 3w = 12 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 2 & -1 & 3 & -1 & 2 \\ 1 & 4 & 0 & 3 & 12 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 3 & -1 & 2 & 7 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & -3 & 1 & -3 & -8 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right)$$

echelon form

$$\begin{aligned} \underline{x} + y + z + w &= 5 \\ \underline{-3y} + z - 3w &= -8 \\ -\underline{w} &= -1 \end{aligned}$$

$$\boxed{\begin{array}{ll} x, y, w: & \text{basic} \\ z: & \text{free} \end{array}}$$

infinitely many solutions, one degree of freedom

V is a line in $\mathbb{R}^4$

Back substitution:

$-w = -1 \Rightarrow \underline{w = 1}$

$-3y + z - 3w = -8$

$-3y = -8 - z + 3w = -8 - z + 3$

$\dfrac{-3y}{-3} = \dfrac{-5 - z}{-3} \Rightarrow \underline{y = \dfrac{5}{3} + \dfrac{1}{3}z}$

$$\begin{cases} x + y + z + w = 5 \\ x = 5 - y - z - w \\ \quad = 5 - (5/3 + 1/3\, z) - z - 1 \\ \quad = \underline{7/3 - 4/3 \cdot z} \end{cases}$$

$\underline{V}$:

$(x, y, z, w) = \underline{\left(\tfrac{7}{3} - \tfrac{4}{3}z, \tfrac{5}{3} + \tfrac{1}{3}z, z, 1\right)}$

where z is free

Theorem:

Any matrix can be transformed into an echelon form (reduced echelon form) using elementary row operations. An echelon form is not unique, but its pivot positions are. (The reduced echelon form is unique).

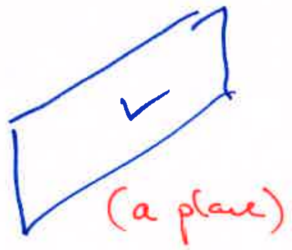
Define:

If a linear system has infinitely many solutions, the number of degrees of freedom is the number of free variables.

Geometric interpretation: degrees of freedom = dimension of the set V of solutions

1 free var:

2 free var.



③ Rank of a matrix and the number of solutions of linear systems

Defn: Let A be any $m \times n$ matrix. The rank of A is the number of pivot positions in A = the number of pivots in an echelon form.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix} \begin{matrix} \\ -4 \\ -2 \end{matrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -12 \\ 0 & -5 & -4 \end{pmatrix} \begin{matrix} \\ \cdot 5 \\ \cdot 7 \end{matrix}$$

$$\to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -35 & -60 \\ 0 & -35 & -28 \end{pmatrix} \begin{matrix} \\ \\ -1 \end{matrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -35 & -60 \\ 0 & 0 & 32 \end{pmatrix}$$

$\text{rk}\, A = 3$ ← $\text{rk}(A)$ = rank of A

Note: $\text{rk}\, A = 0$ if and only if $A = 0$ (a matrix of zeros, the zero matrix)

otherwise $\text{rk}\, A = 1, 2, 3, \ldots$ (a positive integer)

A $m \times n$-matrix: $\text{rk}\, A \leq m$ and $\text{rk}\, A \leq n$

Theorem:

Let us consider any $m \times n$ linear system, with coefficient matrix A and the augmented matrix $(A \mid \underline{b})$. Then we have:

i) there are no solution if and only if $rk(A) \neq rk(A \mid \underline{b})$

ii) If $rk(A) = rk(A \mid \underline{b})$:

$n = rk(A)$ If $\boxed{n - rk(A) = 0}$ then there is one solution

$n > rk(A)$ If $\boxed{n - rkA > 0}$ then there are infinitely many solutions and $\boxed{n - rk(A)}$ degrees of freedom

$\boxed{\dim V = n - rk(A)}$

V = all solutions of the linear system

n: total number of variables
rk(A): number of basic variables
n-rk(A): number of free variables

Homogeneous linear system:

Special case $b_1 = b_2 = \dots = b_m = 0$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}\right\}$$

Trivial solution:

$(x_1, x_2, \dots, x_n) = (0, 0, \dots, 0)$

If there are other solutions, they are called non-trivial solutions.

Theorem:

A homogeneous $m \times n$ linear system has non-trivial solution if and only if $rk\, A < n$.

Worked example of homogeneous system:

$$\begin{aligned} x + 2y - z &= 0 \\ 2x - y + 3z &= 0 \\ 3x + y + 2z &= 0 \end{aligned}$$

homogeneous 3×3 system

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & +2 \end{pmatrix} \begin{array}{l} \\ -2 \\ -3 \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & -5 & 5 \end{pmatrix} \begin{array}{l} \\ \\ -1 \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

echelon form

$\operatorname{rk} A = 2 \Rightarrow n - \operatorname{rk}(A) = 3 - 2 = 1$ degrees of freedom

$\Downarrow$

there are non-trivial solutions

(infinitely many solutions, one degree of freedom)

Explicit computation:

$$\begin{aligned} x + 2y - z &= 0 \\ -5y + 5z &= 0 \end{aligned}$$

$-5y = -5z \Rightarrow y = z$

$x + 2y - z = 0$

$x = -2y + z = -2z + z = -z$

$(x, y, z) = (-z, z, z)$ with z free

Solutions:

A line in $\mathbb{R}^3$ that passes through $(0,0,0)$

(the origin = the trivial solution)