GRA 6035 MATHEMATICS

Problems for Lecture 3

Key problems

Consider the 3-vectors given by

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 4 \\ 3 \\ 9 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Problem 1.
In each case, determine when $\mathbf{w}$ is in the span V, and compute the dimension of V:
a) $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ b) $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ c) $V = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ d) $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$

Problem 2.
For each set of vectors, determine if the vectors are linearly independent:
a) $\{\mathbf{v}_1, \mathbf{v}_2\}$ b) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ c) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ d) $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ e) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

Problem 3.
Find Null(A) for the matrix $A = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{v}_4|\mathbf{v}_5)$; that is, the set of solutions of the homogeneous linear system $A \cdot \mathbf{x} = \mathbf{0}$. Write Null($A$) as the span of a set $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ of linearly independent vectors. What is the value of r?

Problems from the Digital Workbook

Exercise problems	3.1 - 3.12 (full solutions in the workbook)
Exam problems	3.13 - 3.15 (full solutions in the workbook)

Answers to key problems

Problem 1.
a) When $6a - b - 2c = 0$, and $\dim V = 2$ b) For all a, b, c, and $\dim V = 3$ c) When $b + c - 3a = 0$, and $\dim V = 2$
d) For all a, b, c, and $\dim V = 3$

Problem 2.
a) Yes b) Yes c) Yes d) No e) No

Problem 3.
We have that $\text{Null}(A) = \text{span}(\mathbf{w}_1, \mathbf{w}_2)$ with $r = 2$ and

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ -5 \\ 3 \\ 6 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ -5 \\ -9 \\ 0 \\ 6 \end{pmatrix}$$

Solutions: Key problems Lecture 3

1. a) $(\underline{v}_1 | \underline{v}_2 | \underline{w}) = \left(\begin{array}{cc|c} -1 & 3 & a \\ 2 & 0 & b \\ -4 & 9 & c \end{array}\right) \begin{matrix} \\ \xleftarrow{} 2 \\ \xleftarrow{} -4 \end{matrix} \rightarrow \left(\begin{array}{cc|c} -1 & 3 & a \\ 0 & 6 & b+2a \\ 0 & -3 & c-4a \end{array}\right) \begin{matrix} \\ \\ \xleftarrow{} \frac{1}{2} \end{matrix}$

$\rightarrow \left(\begin{array}{cc|c} -1 & 3 & a \\ 0 & 6 & b+2a \\ 0 & 0 & c+\frac{1}{2}b-3a \end{array}\right)$

$\underline{w}$ in $V = \text{span}(\underline{v}_1, \underline{v}_2)$ when there are solutions

$\Updownarrow$

$c + \frac{1}{2}b - 3a = 0$

$\underline{2c + b - 6a = 0}$

$\dim V = \underline{\underline{2}}$ number of (= pivot pos.) in A

$$\begin{array}{c} \begin{array}{cccc} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{w} \end{array} \end{array}$$

b) $\left(\begin{array}{ccc|c} -1 & 3 & 1 & a \\ 2 & 0 & 2 & b \\ -4 & 9 & 1 & c \end{array}\right) \begin{matrix} \\ \xleftarrow{} 2 \\ \xleftarrow{} -4 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} -1 & 3 & 1 & a \\ 0 & 6 & 4 & b+2a \\ 0 & -3 & -3 & c-4a \end{array}\right) \begin{matrix} \\ \\ \xleftarrow{} 1/2 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|c} -1 & 3 & 1 & a \\ 0 & 6 & 4 & b+2a \\ 0 & 0 & -1 & c+\frac{1}{2}b-3a \end{array}\right)$

$\underline{w}$ in $V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ when there are sol's

$\Updownarrow$

for all a, b, c

$\dim V = \text{rk}(\underline{v}_1 | \underline{v}_2 | \underline{v}_3) = \underline{3}$

$$\begin{array}{cccc} \underline{v}_2 & \underline{v}_3 & \underline{v}_4 & \underline{w} \end{array}$$

c) $\left(\begin{array}{ccc|c} 3 & 1 & 2 & a \\ 0 & 2 & -1 & b \\ 9 & 1 & 7 & c \end{array}\right) \begin{matrix} \\ \\ \xleftarrow{} -3 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & a \\ 0 & 2 & -1 & b \\ 0 & -2 & 1 & c-3a \end{array}\right) \begin{matrix} \\ \\ \xleftarrow{} 1 \end{matrix}$

$\rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & a \\ 0 & 2 & -1 & b \\ 0 & 0 & 0 & c+b-3a \end{array}\right)$

$\underline{w}$ in $V = \text{span}(\underline{v}_2, \underline{v}_3, \underline{v}_4)$ when there are sol's

$\Updownarrow$

$c + b - 3a = 0$

$\dim V = \underline{\underline{2}}$

d) $\left(\begin{array}{cccc|c} -1 & 3 & 1 & 2 & a \\ 2 & 0 & 2 & -1 & b \\ -4 & 9 & 1 & 7 & c \end{array}\right) \xrightarrow{2,\;-4} \left(\begin{array}{cccc|c} -1 & 3 & 1 & 2 & a \\ 0 & 6 & 4 & 3 & b+2a \\ 0 & -3 & -3 & -1 & c-4a \end{array}\right) \xrightarrow{\frac{1}{2}}$

$\rightarrow \left(\begin{array}{cccc|c} -1 & 3 & 1 & 2 & a \\ 0 & 6 & 4 & 3 & b+2a \\ 0 & 0 & -1 & 1/2 & c+\frac{1}{2}b-3a \end{array}\right)$

$\underline{w}$ in $V = \text{span}(\underline{v_1}, \dots, \underline{v_4})$ when there are sol's

$\Updownarrow$

for all a, b, c

$\dim V = \underline{\underline{3}}$

2. a) Yes, pivot pos. in each col. (see 1 a))

b) Yes, — " —

d) No, no pivot pos. in $\underline{v_3}$ col. $\Rightarrow$ $\underline{v_3}$ is lin. comb. of $\underline{v_1}, \underline{v_2}$.

e) No, no pivot pos in $\underline{v_4}$ col. $\Rightarrow$ $\underline{v_4}$ is lin. comb. of $\underline{v_1}, \underline{v_2}, \underline{v_3}$.

c) $\begin{array}{c} \begin{array}{ccc} \underline{v_1} & \underline{v_2} & \underline{v_4} \end{array} \\ \begin{pmatrix} -1 & 3 & 2 \\ 2 & 0 & -1 \\ -4 & 9 & 7 \end{pmatrix} \end{array} \xrightarrow{2,\;-4} \begin{pmatrix} -1 & 3 & 2 \\ 0 & 6 & 3 \\ 0 & -3 & -1 \end{pmatrix} \xrightarrow{1/2} \begin{pmatrix} -1 & 3 & 2 \\ 0 & 6 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$

$\underline{\text{Yes}}$, pivot pos. in each col.

3. $A = (\underline{v_1} | \underline{v_2} | \underline{v_3} | \underline{v_4} | \underline{v_5})$

Null(A): Solutions of $A\underline{x} = \underline{0}$

$\begin{array}{c} \begin{array}{ccccc} \underline{v_1} & \underline{v_2} & \underline{v_3} & \underline{v_4} & \underline{v_5} \end{array} \\ \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 2 & 0 & 2 & -1 & 3 & 0 \\ 4 & 9 & 1 & 7 & 9 & 0 \end{array}\right) \end{array} \xrightarrow{2,\;-4} \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 0 & 6 & 4 & 3 & 11 & 0 \\ 0 & -3 & -3 & -1 & -7 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}}$

$\rightarrow \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 0 & 6 & 4 & 3 & 11 & 0 \\ 0 & 0 & -1 & 1/2 & -3/2 & 0 \end{array}\right)$

two free var (x_4, x_5), inf. many solutions

$\Downarrow$

$\text{Null}(A) = \text{span}(\underline{w_1}, \underline{w_2})$

$$
\begin{aligned}
-\underline{x_1} + 3x_2 + x_3 + 2x_4 + 4x_5 &= 0 \\
6\underline{x_2} + 4x_3 + 3x_4 + 11x_5 &= 0 \\
-\underline{x_3} + \tfrac{1}{2}x_4 - \tfrac{3}{2}x_5 &= 0
\end{aligned}
$$

Backwards Substitution:

$$-x_3 = -\frac{1}{2}x_4 + \frac{3}{2}x_5 \Rightarrow x_3 = \underline{\frac{1}{2}x_4 - \frac{3}{2}x_5}$$

$$6x_2 = -4x_3 - 3x_4 - 11x_5 = -4\left(\frac{1}{2}x_4 - \frac{3}{2}x_5\right) - 3x_4 - 11x_5$$

$$= -5x_4 - 5x_5 \Rightarrow x_2 = \underline{-\frac{5}{6}x_4 - \frac{5}{6}x_5}$$

$$-x_1 = -3x_2 - x_3 - 2x_4 - 4x_5 = -3\left(-\frac{5}{6}x_4 - \frac{5}{6}x_5\right) - \left(\frac{1}{2}x_4 - \frac{3}{2}x_5\right) - 2x_4 - 4x_5$$

$$= \frac{5}{2}x_4 - \frac{1}{2}x_4 - 2x_4 + \frac{5}{2}x_5 + \frac{3}{2}x_5 - 4x_5 = 0 \Rightarrow x_1 = \underline{0}$$

Solutions: (in vector form)

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}
=
\begin{pmatrix} 0 \\ -5/6\, x_4 - 5/6\, x_5 \\ 1/2\, x_4 - 3/2\, x_5 \\ x_4 \\ x_5 \end{pmatrix}
=
\begin{pmatrix} 0 \\ -5/6\, x_4 \\ 1/2\, x_4 \\ x_4 \\ 0 \end{pmatrix}
+
\begin{pmatrix} 0 \\ -5/6\, x_5 \\ -3/2\, x_5 \\ 0 \\ x_5 \end{pmatrix}
$$

$$
= x_4 \cdot \underbrace{\begin{pmatrix} 0 \\ -5/6 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}}_{\text{"}\underline{w_1}\text{"}}
+ x_5 \underbrace{\begin{pmatrix} 0 \\ -5/6 \\ -3/2 \\ 0 \\ 1 \end{pmatrix}}_{\text{"}\underline{w_2}\text{"}}
= \frac{x_4}{6} \cdot \underbrace{\begin{pmatrix} 0 \\ -5 \\ 3 \\ 6 \\ 0 \end{pmatrix}}_{\text{"}\underline{w_1}'\text{"}}
+ \frac{x_5}{6} \underbrace{\begin{pmatrix} 0 \\ -5 \\ -9 \\ 0 \\ 6 \end{pmatrix}}_{\text{"}\underline{w_2}'\text{"}}
$$

$$\mathrm{Null}(A) = \underline{\underline{\mathrm{span}(\underline{w_1}, \underline{w_2})}} = \underline{\underline{\mathrm{span}(\underline{w_1}', \underline{w_2}')}}$$

both alternatives
can be used