Plan:

1. Final Exam 01/2017: Problem 3-4
2. Workbook 7.1, 8.13
3. Key problems 7.3b, 7.4b, 8.3, 9.1, 9.3

Final exam 01/2017

3. $f = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2$

a) $\left.\begin{array}{l} f'_x = -4x + 2y - 2z = 0 \\ f'_y = 2x - 4y + 4z = 0 \\ f'_z = -2x + 4y - 4z = 0 \end{array}\right\}$ $H(f) = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

$D_1 = -4 < 0$

$D_2 = 16 - 4 = 12 > 0$

$|H(f)| = D_3 = -2 \cdot 0 - 4 \cdot (-12) + (-4) \cdot 12$
$= 48 - 48 = 0$

Rec: $\operatorname{rk} H(f) = 2$

f is concave $\Longleftarrow$ $H(f)$ negative semidefin. for all x, y, z

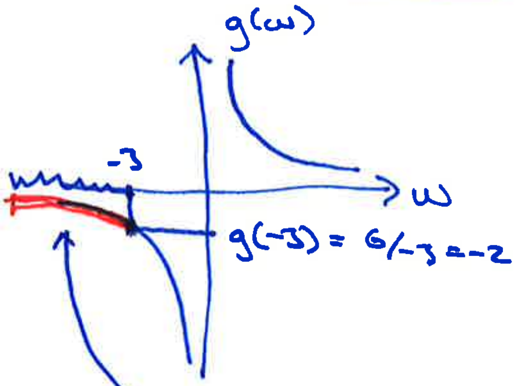
f is not convex (since $D_1 < 0$)

b) Since f is concave, any stationary pt. is global max

$(0,0,0)$ is a stationary pt $\Rightarrow$ global max $\Rightarrow$ $f_{max} = f(0,0,0)$
$= -3$

c) $g(x,y,z) = \dfrac{6}{f(x,y,z)} = \dfrac{6}{w}$, $w = f(x,y,z)$

$$\boxed{\begin{array}{l} g: \text{composite fn.} \\ (x,y,z) \longrightarrow w = f(x,y,z) \longrightarrow g = \dfrac{6}{w} \end{array}}$$



From a) b):

max $w = -3$

attainable values of $g(w)$

Max: there is no max $g \to 0$ when $w \to -\infty$

Min: $g_{min} = -2$

(Alt: $g(w) = 6/w \Rightarrow g'(w) = (6w^{-1})' = 6\cdot(-1)w^{-2} = \frac{-6}{w^2} < 0$)

4. max $f(x,y,z) = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2$ when $x + y - z \geq 2 \Leftrightarrow -x - y + z \leq -2$

a) $\mathcal{L} = f(x,y,z) - \lambda \cdot (-x - y + z)$

FOC
$$\begin{cases} \mathcal{L}'_x = -4x + 2y - 2z + \lambda = 0 \\ \mathcal{L}'_y = 2x - 4y + 4z + \lambda = 0 \\ \mathcal{L}'_z = -2x + 4y - 4z - \lambda = 0 \end{cases}$$

Kuhn-Tucker cond: FOC + C + CSC

C $\{ \; -x - y + z \leq -2$

CSC $\{ \; \lambda \geq 0, \quad \lambda(-x - y + z + 2) = 0$

$\Leftrightarrow$ a) $-x - y + z = -2, \lambda \geq 0$ or b) $-x - y + z < -2, \lambda = 0$

b) Solve the KT problem:

i) $-x - y + z < -2, \lambda = 0$: (x,y,z) stationary pt of f

$$\begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix} \to \begin{pmatrix} -4 & 2 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} -4 & 2 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

z free

$-3y + 3z = 0 \Rightarrow y = z$

$-4x + 2y - 2z = 0 \Rightarrow -4x = 0 \Rightarrow x = 0$

~~Cand. pts: $(0, z, z; 0)$ $f = -3$ (z free)~~

$x = 0, y = z$

C: $0 - z + z < -2$

$0 < -2$

not candidate pts.

ii) $-x - y + z = -2, \lambda \geq 0$:

$$\begin{aligned} -4x + 2y - 2z + \lambda &= 0 \\ 2x - 4y + 4z + \lambda &= 0 \\ -2x + 4y - 4z - \lambda &= 0 \\ -x - y + z &= -2 \end{aligned}$$

$$\left(\begin{array}{cccc|c} -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \\ -1 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} -1 & -1 & 1 & 0 & -2 \\ 2 & -4 & 4 & 1 & 0 \\ -2 & 4 & -4 & -1 & 0 \\ -4 & 2 & -2 & 1 & 0 \end{array}\right) \begin{matrix} 2 \\ -2 \\ -4 \end{matrix} \to \left(\begin{array}{cccc|c} -1 & -1 & 1 & 0 & -2 \\ 0 & -6 & 6 & 1 & -4 \\ 0 & 6 & -6 & -1 & 4 \\ 0 & 6 & -6 & 1 & 8 \end{array}\right) \begin{matrix} 1 \\ 1 \end{matrix}$$

$$\to \left(\begin{array}{cccc|c} -1 & -1 & 1 & 0 & -2 \\ 0 & -6 & 6 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right)$$

z free

$-6y + 6z + \lambda = -4 \Rightarrow \frac{-6y}{-6} = \frac{-4-2-6z}{-6}$

$2\lambda = 4 \Rightarrow \underline{\lambda = 2}$ $\qquad y = \underline{1+z}$

$-x - y + z = -2$

$\frac{-x}{-1} = -2 + (1 + \cancel{z}) - \cancel{z} = \frac{-1}{-1}$

$\underline{x = 1}$

<u>cand pts:</u>

$(x, y, z; \lambda) = \underline{(1, 1+z, z; 2)}$ $\qquad \lambda = 2 \geq 0$ vok.

(z free)

$f = f(1, 1+z, z)$

<u>SOC:</u> check $(1, 1+z, z; 2)$

$h(x,y,z) = h(x,y,z; 2) = f(x,y,z) - 2(-x-y+z)$

$= f(x,y,z) + 2x + 2y - 2z$

$H(h) = H(f)$ since $\left(\begin{array}{c} 2x+2y-2z \\ \text{is linear} \end{array}\right)$ + f concave (from 3a)

$\Rightarrow h$ concave

$\Rightarrow (x,y,z) = (1, 1+z, z)$ is <u>max</u> pt for all z

$f_{max} = f(1,1,0) = \underline{\underline{-5}}$

c) Parametrized KT-problem

max $f(x,y,z)$ when $ax+y-z \geq 2$

$-ax-y+z \leq -2$

$a=1$: $f^*(1) = -5$

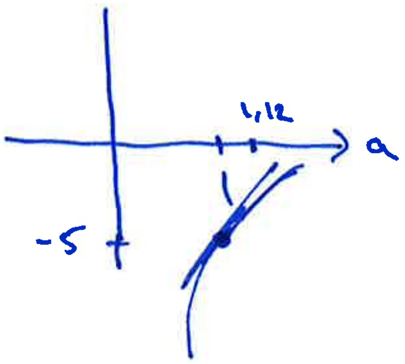
$x^*(1) = 1 \qquad \lambda^*(1) = 2$

$y^*(1) = 1+z$

$z^*(1) = z$

$$L = f(x,y,z) \underbrace{-\lambda\cdot(-ax-y+z+2)}_{+a\lambda x}$$

Envelope thm: $$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}\left(x^*(a), y^*(a), z^*(a); \lambda^*(a)\right)$$



Slope of tangent line in $a=1$:

$$\frac{\partial L}{\partial a} = \lambda x$$

$$\frac{df^*(a)}{da}(a=1) = \lambda^*(1)\cdot x^*(1) = 2\cdot 1 = \underline{\underline{2}}$$

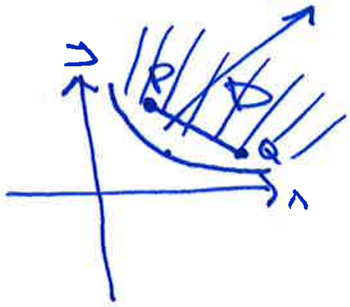
Estimate: $$f^*(1{,}12) \simeq \underset{-5}{f^*(1)} + \underset{0.12}{\Delta a} \cdot \underset{2}{\frac{df^*(a)}{da}}$$

$$= -5 + 0.12\cdot 2 = -5 + 0.24 = \underline{\underline{-4.76}}$$

(2) Workbook.

7.1. v) $x \geq 0, y \geq 0$

$xy \geq 1$



$xy = 1$: $y = 1/x$

$xy > 1$: $y > 1/x$

D is closed ($\geq, \leq$)

D is not bounded

(no upper bound on x/y)

D is convex (D is the set of points over the graph of $y = 1/x$, which is convex

$y' = -1/x^2 \quad y'' = 2/x^3 > 0$)

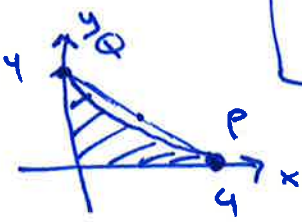
vi) $\sqrt{x} + \sqrt{y} \leq 2$

Closed: ($\leq$)

Bounded: Yes

$0 \leq x \leq 4$

$0 \leq y \leq 4$



Convex:

$P = (4,0)$, $Q = (0,4)$ } midpoint. $(2,2)$

$\sqrt{2} + \sqrt{2} = 2\sqrt{2} \simeq 2.8$

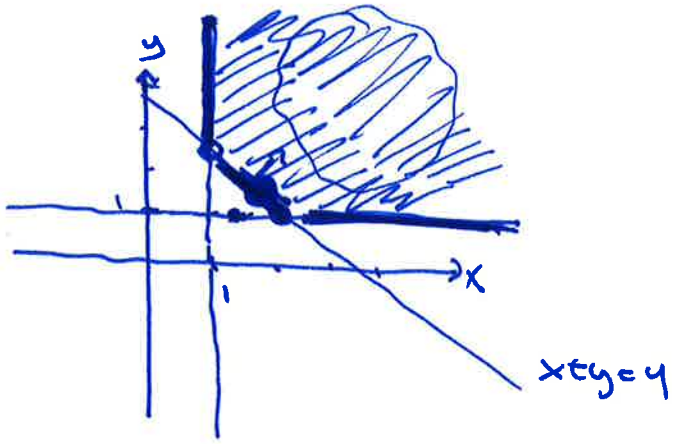
$(2,2)$ not in D

$\Downarrow$

D not convex

(Final exam 12/2012 Q.5)

8.13 $\max \ln(x^2y)-x-y$ when $\begin{cases} x+y \geq 4 \\ x \geq 1 \\ y \geq 1 \end{cases}$



$x+y=4$

$y=4-x$

$\max \ln(x^2y)-x-y$ when $\begin{cases} -x-y \leq -4 \\ -x \leq -1 \\ -y \leq -1 \end{cases}$

$\qquad \| $

$2\ln x + \ln y$

$\mathcal{L} = 2\ln x + \ln y - x - y - \lambda_1(-x-y) - \lambda_2(-x) - \lambda_3(-y)$

$= 2\ln x + \ln y - x - y + \lambda_1(x+y) + \lambda_2 x + \lambda_3 y$

$\mathcal{L}'_x = \frac{2}{x} - 1 + \lambda_1 + \lambda_2 = 0$

$\mathcal{L}'_y = \frac{1}{y} - 1 + \lambda_1 + \lambda_3 = 0$

$x+y \geq 4$

$x, y \geq 1$

$\lambda_1 \geq 0 \quad \lambda_1(x+y-4)=0$

$\lambda_2 \geq 0 \quad \lambda_2(x-1)=0$

$\lambda_3 \geq 0 \quad \lambda_3(y-1)=0$

$\underline{x=1}$: $\underbrace{2-1}_{1} + \underbrace{\lambda_1+\lambda_2}_{\geq 0} = 0$ not possible $\Rightarrow$ $\boxed{\begin{matrix} x>1 \\ \lambda_2=0 \end{matrix}}$

$\underline{y=1}$: $\underbrace{1-1}_{0} + \underbrace{\lambda_1+\lambda_3}_{\geq 0} = 0 \Rightarrow \lambda_1=\lambda_3=0$

$\Rightarrow \frac{2}{x} - 1 + 0 + 0 = 0 \Rightarrow x=2$

not possible $\Rightarrow$ $\boxed{\begin{matrix} y>1 \\ \lambda_3=0 \end{matrix}}$

$x>1,\ y>1$

$\lambda_2=\lambda_3=0$

$x+y=4$: FOC $\begin{cases} \frac{2}{x} - 1 + \lambda_1 = 0 \\ \frac{1}{y} - 1 + \lambda_1 = 0 \end{cases}$

$\lambda_1 = 1 - \frac{2}{x}$

$\Downarrow$

$\frac{1}{y} \cancel{-1} + (\cancel{1} - \frac{2}{x}) = 0$

$\frac{1}{y} = \frac{2}{x}$

$x = 2y$

$x+y=4$

$2y+y=4$

$3y=4$

$x = 8/3$

$y = 4/3 \Leftarrow$

$\lambda_1 = 1 - \frac{2\cdot 3}{8/3 \cdot 3}$

$= 1 - \frac{6}{8} = \frac{2}{8} = \frac{1}{4}$

Cand pt:

$(x,y;\lambda_1,\lambda_2,\lambda_3) = \Leftarrow$

$(8/3, 4/3; 1/4, 0, 0)$

$f = 2\ln(8/3) + \ln(4/3) - \frac{8}{3} - \frac{4}{3}$

SOC: $h(x,y) = L(x,y;1/4,0,0)$

$= 2\ln x + \ln y - x - y - \frac{1}{4}(-x-y) - 0\cdot(-x) - 0(-y)$

$= 2\ln x + \ln y - x - y + 1/4\,x + 1/4\,y$

$h'_x = \frac{2}{x} - 1 + 1/4$

$h'_y = \frac{1}{y} - 1 + 1/4$

$H(h) = \begin{pmatrix} -2/x^2 & 0 \\ 0 & -1/y^2 \end{pmatrix}$

$D_1 = -2/x^2 < 0$ for all x,y

$D_2 = 2/x^2y^2 > 0$ — " —

$\Downarrow$

h concave

$(x,y) = (8/3, 4/3)$ is max $\Leftarrow$ SOC

Key problem 8.3 b)

max $f = xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

$L = xz + yw - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$

FOC: 1) $L'_x = z - \lambda_1 \cdot 2x = 0$
2) $L'_y = w - \lambda_1 \cdot 2y = 0$
3) $L'_z = x - \lambda_2 \cdot 8z = 0$
4) $L'_w = y - \lambda_2 \cdot 18w = 0$

C: $x^2+y^2 \le 1$
$4z^2+9w^2 \le 36$

CSC: $\lambda_1 \ge 0$ $\lambda_1(x^2+y^2-1) = 0$
$\lambda_2 \ge 0$ $\lambda_2(4z^2+9w^2-36) = 0$

FOC: (1) $z = 2\lambda_1 x \Rightarrow$ (3) $x - 8\lambda_2 \cdot (2\lambda_1 x) = 0$
$x - 16\lambda_1\lambda_2 x = 0$
$x(1-16\lambda_1\lambda_2) = 0$
$x=0$ or $\lambda_1\lambda_2 = 1/16$

(2) $w = 2\lambda_1 y \Rightarrow$ (4) $y - 18\lambda_2(2\lambda_1 y) = 0$
$y - 36\lambda_1\lambda_2 y = 0$
$y(1-36\lambda_1\lambda_2) = 0$
$y=0$ or $\lambda_1\lambda_2 = 1/36$

a) $x=0, y=0$
b) $x=0, \lambda_1\lambda_2 = 1/36$
c) $\lambda_1\lambda_2 = 1/16, y=0$

(a) $x=0, y=0$: $z=0, w=0$
$\lambda_1=0, \lambda_2=0$

$\Rightarrow (0,0,0,0;0,0)$ $f=0$

b) $x=0,\ \lambda_1\lambda_2=\frac{1}{36}$: $z=0$, $\lambda_1>0, \lambda_2>0$

$\Downarrow$

$x^2+y^2=1$, $4z^2+9w^2=36$

$y^2=1$ $\quad 9w^2=36$

$y=\pm 1$ $\quad w^2=4$

$w=\pm 2$

$(x,y,z,w;\lambda_1,\lambda_2)=$

$(0,\pm 1,0,\pm 2;\lambda_1,\lambda_2)$

(2): $\lambda_1=\frac{w}{2y}=\frac{\pm 2}{2\cdot(\pm 1)}=1$

w, y same sign

$\Downarrow$

$(0,+1,0,+2;1,1/36)$, $f=2$

$(0,-1,0,-2;1,1/36)$ $f=2$

c) In the same way: $(1,0,3,0;3/2,1/24)$ $f=3$

$(-1,0,-3,0;3/2,1/24)$ $f=3$

$\uparrow$

Best candidate points

SOC: h not concave $\Rightarrow$ no conclusion from SOC.

EVT: $x^2+y^2\le 1$ $\quad -1\le x\le +1$ $\quad$ D is bounded

$4z^2+9w^2\le 36$ $\quad -1\le y\le +1$ $\quad \Downarrow$ EVT

$-3\le z\le 3$ $\quad$ there is a maximum

$-2\le w\le 2$

Check NDCQ: satisfied at all adm points

$\Downarrow$

$f_{max}=3$ at $(1,0,3,0),(-1,0,-3,0)$