Plan:

Review Lecture 1

1. Matrices and matrix algebra
2. Determinants and minors
3. Rank and linear systems using minors.

Review:

You should:

- be able to solve linear systems using Gaussian elimination
- understand linear systems
- know what the rank of a matrix is

$$\text{rk}(A) = \#\text{pivot positions in } A$$

pivot positions are the important result of Gaussian elimination

$\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 7 \end{array}\right) \to$ no sol's

$\left(\begin{array}{cccc|c} 1 & 7 & 3 & 4 & 9 \\ 0 & 0 & 1 & -1 & 5 \end{array}\right) \to y, w$ free
inf. many solutions

① Matrices and matrix algebra

An $m \times n$-matrix A is a rectangular array of numbers with m rows, n columns

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix}$$

a_{ij}: entry in A in pos. (i,j) = row i, col. j.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 4 & -1 \end{pmatrix}$$

Operations on matrices:

- addition / subtraction: $\left.\begin{matrix} A+B \\ A-B \end{matrix}\right\}$ defined if A, B have the same size, computed position by position

Ex:

$$A+B = \begin{pmatrix} 1 & 2 & 7 \\ 0 & 4 & -1 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 4 & -1 \end{pmatrix}$$

zero matrix: $0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\boxed{A + 0 = A}$

- scalar multiplication: $r \cdot A$ r number (scalar) A matrix, computed position by position

Ex:

$$2 \cdot A = 2 \cdot \begin{pmatrix} 1 & 2 & 7 \\ 0 & 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 14 \\ 0 & 8 & -2 \end{pmatrix}$$

– matrix multiplication: $\underset{m\times n}{A}\cdot\underset{n\times p}{B}$

defined if #cols(A) = #rows(B), the result is a $m\times p$-matrix

Ex:

$$\underset{2\times 2}{A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}} \cdot \underset{2\times 3}{B = \begin{pmatrix} 1 & 7 & 4 \\ 0 & -1 & 3 \end{pmatrix}} = \underset{2\times 3}{\begin{pmatrix} 1 & 5 & 10 \\ 0 & 1 & -3 \end{pmatrix}}$$

$(AB)_{11} = 1\cdot 1 + 2\cdot 0 = 1$

$(AB)_{12} = 1\cdot 7 + 2\cdot(-1) = 5$

$\boxed{AB \neq BA}$

$\underset{2\times 3}{B}\cdot\underset{2\times 2}{A}$ is not defined

Identity matrix: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$$\boxed{\begin{matrix} A\cdot I = A \\ I\cdot A = A \end{matrix}}$$

$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

$$\boxed{\begin{matrix} (A+B)\cdot C \\ = AC + BC \end{matrix}}$$

– transpose: $\underset{m\times n}{A} = (a_{ij}) \rightsquigarrow \underset{n\times m}{A^T} = (a_{ji})$

Ex: $A = \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 7 \end{pmatrix}}$ $A^T = \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 7 \end{pmatrix}}$

Defn: A is called symmetric if $A^T = A$.

$A = \underset{3\times 3}{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 7 & 1 & -1 \end{pmatrix}}$ $A^T = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 7 \\ 2 & 1 & 1 \\ 4 & 3 & -1 \end{pmatrix}}$

$$\boxed{(AB)^T = B^T\cdot A^T}$$

A <u>square</u> matrix is a matrix A with #rows(A) = #cols(A)

<u>Powers:</u>

$$\underset{n\times n}{A} \rightsquigarrow A^2 = A\cdot A$$

$$A^3 = A^2 \cdot A$$

$$\vdots$$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

<u>Diagonal matrix:</u>

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

$n\times n$-matrix with zeros in all positions outside the diagonal.

<u>Ex:</u>

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{17} = \begin{pmatrix} 2^{17} & 0 \\ 0 & 3^{17} \end{pmatrix}$$

Powers of A are hard to compute, but it is much easier for <u>diagonal matrices</u>

Inverses: An *inverse* of the $(n\times n)$ matrix A is a matrix A^{-1} with the property that

$$A\cdot A^{-1} = I \text{ and } A^{-1}\cdot A = I$$

Facts:
- If the inverse exists, then it is unique
- A is called invertible if A^{-1} exists.

$$\boxed{A \text{ invertible} \iff |A| \neq 0}$$

(only square matrices can be invertible)

The case $n=2$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad |A| = ad-bc$$

$$A^{-1} = \frac{1}{ad-bc}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \text{ if } ad-bc \neq 0$$

A^{-1} does not exist if $ad-bc=0$.

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}} \leftarrow \text{if } A, B \text{ are invertible (of any size)}$$

Linear systems and inverses:

$$\left.\begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{matrix}\right\} \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$\longrightarrow$ $A\cdot \underline{x} = \underline{b}$ (linear system in matrix form)

If A is invertible:

$$A\cdot\underline{x} = \underline{b}$$

$$A^{-1}\cdot A\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\cdot\underline{b}$$

One unique solution

Methods for computing A^{-1} when $n > 2$:

(1) If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$, where the

adjungated matrix $\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$

C_{ij} = cofactor in pos. (i,j) ; see under determinants.

(2) Using Gauss:

$$\underset{n\times n}{A} \longrightarrow (A \mid I_n) \xrightarrow{\text{elementary row op.}} \cdots \rightarrow (B \mid C)$$

Construct a matrix consisting of A and I_n

$(B \mid C)$: echelon form

Thm: A invertible $\Leftrightarrow$ B has a pivot pos. in every col.
In that case, further row operations will bring the echelon form to reduced echelon form

$$(B \mid C) \xrightarrow{\text{further el. row op.}} \cdots \rightarrow (I_n \mid A^{-1})$$

read off A^{-1}

② Determinants

$\underset{n\times n}{A} \rightsquigarrow \det(A) = |A|$ (a number)

$$\boxed{\begin{array}{c} A \text{ invertible} \\ \Updownarrow \\ |A| \neq 0 \end{array}}$$

How to compute determinants:

n=2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$

Method I: Cofactor expansion

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 0 & 4 & 7 \end{pmatrix}$

Cofactors: $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$\begin{aligned} |A| &= 1 \cdot C_{11} + 1 \cdot C_{21} + 0 \cdot C_{31} \\ &= 1 \cdot (+1) \cdot M_{11} + 1 \cdot (-1) \cdot M_{21} + 0 \cdot (+1) \cdot M_{31} \\ &= +1 \cdot (-15) - 1 \cdot 2 + 0 \cdot M_{31} = \underline{\underline{-17}} \end{aligned}$$

M_{ij} = minor in pos. (i,j)
= determinant of submatrix you get by deleting row i, col j.

$M_{11} = \begin{vmatrix} -1 & 2 \\ 4 & 7 \end{vmatrix} = -15 \qquad M_{21} = \begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} = 2$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 0 & 4 & 7 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \\ 0 & 4 & 7 \end{pmatrix}$

Facts: * Cofactor expansion along any row or column gives the same result, $\det(A)$.

* algebra rules for determinants:

$$\boxed{\begin{aligned} |A^T| &= |A| \\ |AB| &= |A|\cdot|B| \\ |rA| &= r^n\cdot|A| \quad \text{if } A \text{ is } n\times n \end{aligned}}$$

Ex:

$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = -1\cdot\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -1\cdot(-1\cdot(1-1)) + 1\cdot 0 = \underline{\underline{0}}$$

Method 2: Gaussian elimination

$$\begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \begin{matrix} \\ \\ \leftarrow \\ \\ \end{matrix}\Big]\cdot(-1) \;\to\; = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} = 0$$

Note: Cofactor expansion is a general method (can be used for any determinant). The following method works only for $n=3$:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 1 & 3 & 9 & 1 & 3 \end{vmatrix} = \left\{\begin{matrix} 1\cdot 2\cdot 9 + 1\cdot 4\cdot 1 + 1\cdot 1\cdot 3 \\ -1\cdot 2\cdot 1 - 3\cdot 4\cdot 1 - 9\cdot 1\cdot 1 \end{matrix}\right\} = \left\{\begin{matrix} 18+4+3 \\ -2-12-9 \end{matrix}\right\} = \underline{2}$$

Fact: If $A \to B$ is an elementary row operation, then:

* If the operation is to switch two rows, then $|B| = -|A|$.
* If the operation is to multiply a row by $c \neq 0$, then $|B| = c \cdot |A|$
* If the operation is to add a multiple of one row to another row, then $|B| = |A|$.

Note: * If a matrix A has two equal rows, or two equal columns, then $|A| = 0$.

* If A is $n \times n$: $\boxed{|A| \neq 0 \iff A \text{ has a pivot position in every column.}}$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = E$$

$$|A| \neq 0 \iff |E| \neq 0$$

$$A = \begin{pmatrix} 1 & 7 & 4 \\ 2 & 14 & 9 \\ 3 & 21 & -5 \end{pmatrix} \to \cdots \to \begin{pmatrix} 1 & 7 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|A| = 0 \iff |E| = 0$$

Upper triangular matrices:

An upper triangular matrix is a square matrix on this form:

$$A = \begin{pmatrix} d_1 & * & * & & * \\ 0 & d_2 & * & & * \\ 0 & 0 & d_3 & & * \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$\boxed{|A| = d_1 \cdot d_2 \cdot \ldots \cdot d_n}$$

Formula for determinant of a triangular matrix.

All echelon forms are upper triangular.

Pivot in each column: $d_1 \neq 0, d_2 \neq 0, \ldots$ are the pivots

$$\Updownarrow$$

$$|A| \neq 0$$

Some columns without pivots:

Ex: $\begin{pmatrix} 1 & 0 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $|A| = 0 = 1 \cdot 0 \cdot 0$

$$\boxed{|A| \neq 0 \iff \text{there is a pivot position in each column of } A.}$$

③ Minors, rank and linear systems

Let A be an $m \times n$-matrix. An **r-minor** in A is a determinant of an $r \times r$-submatrix of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{pmatrix}$

2-minors: (maximal minors)

$$M_{12,12} = \begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = \underline{-15} \neq 0$$

$$M_{12,23} = \begin{vmatrix} 2 & 4 \\ -1 & 0 \end{vmatrix} = 4$$

$$M_{12,13} = \begin{vmatrix} 1 & 4 \\ 7 & 0 \end{vmatrix} = -28$$

1-minors:

$$M_{1,1} = 1$$

$$M_{1,2} = 2$$

$$\vdots$$

Rank: The rank of A is the order of a maximal non-zero minor.

$$\mathrm{rk}(A) = \mathrm{rk}\begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \end{pmatrix} = 2$$

since there is a 2-minor that is non-zero, and no bigger non-zero minors.

Notation:

$M_{12,23}$ means:

Keep rows 1,2 (delete the rest)
Keep col's 2,3 (delete the rest)

And take the determinant of the submatrix that remains.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 8 & 9 \\ 3 & 10 & 13 \end{pmatrix}$

$|A| = M_{123,123}$ (maximal minor) 3-minor

$= 1(104-90) - 2(26-27) + 4(20-24)$

$= 14 + 2 - 16 = 0$

All 3-minors are zero : $\operatorname{rk} A < 3$

$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 8 & 9 \\ 3 & 10 & 13 \end{pmatrix}$

2-minors:

The first 2-minor (first = furthest left/up), there are many others. No need to compute all of them.

$M_{12,12} = \begin{vmatrix} 1 & 2 \\ 2 & 8 \end{vmatrix} = 8 - 4 = 4 \neq 0$

one 2-minor is non-zero : $\operatorname{rk} A = 2$

Pivot positions: 1 8 are pivot positions (= where the pivots would end up if you compute an echelon form) since $M_{12,12} \neq 0$

13 not pivot position since $|A| = 0$.

Ex: Find $\operatorname{rk} \begin{pmatrix} a & 1 & 7 \\ 1 & 0 & 4 \\ 2 & 1 & 0 \end{pmatrix}$

3-minors: $|A| = 2 \cdot 4 - 1 \cdot (4a - 7)$

$= 8 - 4a + 7$

$= 15 - 4a$

$|A| = 0$: $15 - 4a = 0$

$a = 15/4$

$a = 15/4$: $|A| = 0 \Rightarrow \operatorname{rk} A < 3$; Check $M_{12,12} = \begin{vmatrix} 15/4 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$ (substitute $a = 15/4$)

$\operatorname{Rk}(A) = 2$

$a \neq 15/4$: $|A| \neq 0 \Rightarrow \operatorname{rk} A = 3$

Special case:

$$\underset{n\times n}{A} \qquad |A| \neq 0 \Rightarrow \text{rk}\, A = n$$
$$|A| = 0 \Rightarrow \text{rk}\, A < n$$

In homogeneous systems:
there are non-trivial sol's
$\Updownarrow$
there are free variables

Linear systems:

Ex:
$$\begin{aligned} x + 2y + z &= 0 \\ 2x - y + 3z &= 0 \\ x + 7y &= 0 \end{aligned}$$

homogeneous
$b_1 = b_2 = b_3 = 0$

$x = y = z = 0$ is trivial sol.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 1 & 7 & 0 \end{pmatrix}$$

3-minors: $|A| = 1 \cdot 15 - 3 \cdot 5 = 0$

$\text{rk}\, A < 3 \Rightarrow$ at least one free variable.

2-minors: $M_{12,12} = 1 \cdot (-1) - 4 = -5 \neq 0$

$\text{rk}\, A = 2 \Rightarrow$ one free variable

z free

$$\begin{aligned} x + 2y &= -z \\ 2x - y &= -3z \end{aligned}$$

I+2II: $5x = -7z$

$x = -7/5 \cdot z$

From II: $2x - y = -3z$

$-y = -3z - 2x$

$y = 3z + 2x$

$= 3z + 2 \cdot (-7/5\, z)$

$= 3z - 14/5\, z = \frac{1}{5} z$

Soln:
$(x, y, z) = (-\frac{7}{5}z, \frac{1}{5}z, z)$ with z free

When $rk(A) = 2$ because of a nonzero minor $M_{12,12} = -5 \neq 0$:
- all eqn's except (1), (2) can be ignored (superfluous) since the minor contains row 1, 2
- Eqn. (1), (2) can be solved for x, y (variable 1, 2) since the minor contains col. 1, 2. All other var's (z) are free.

Problem: Compute $\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}$:

Answer / Solution:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \begin{array}{l} -4 \\ \\ \\ -1 \\ \\ \end{array} = \begin{vmatrix} 0 & -4 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} -4 & 0 & -1 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -2 & 2 & 0 & 0 \end{vmatrix} = -(-2) \cdot \begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ 6 & -2 & 0 \end{vmatrix} - 2 \begin{vmatrix} -4 & -1 & -1 \\ 2 & 1 & -1 \\ 0 & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot 6 \cdot (1-(-1)) - 2 \cdot (-2) \cdot (4-(-2))$$

$$= 12 \cdot 2 + 4 \cdot 6 = 24 + 24 = \underline{\underline{48}}$$