Plan:

(1) About the final exam

(2) Final exam 11/2017

(1) Final exam: 12 questions
- matrix methods
- differential equations
- unconstrained optimization
- constrained optimization

— think about how you write answers to the question
— think about a plan before you start computing

(2) Final exam 11/2017.

1. $f(x,y,z,w) = x^2 + y^2 + 9z^2 + 4w^2 + 6yz - 4yw - 12zw$

a) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{pmatrix} \xrightarrow{-3,\ 2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

echelon form of A

$\operatorname{rk} A = 2$

b) Determinants of A:

$D_1 = 1$

$D_2 = 1$

$D_3 = 0$, $D_4 = 0$ } since $\operatorname{rk} A = 2$

RRC: $D_1, D_2 > 0$ ok
$\operatorname{rk} A = 2$ ok
$\Downarrow$
A positive semidefinite

Revision: RRC (reduced rank criterion)

If A is a symmetric $n \times n$-matrix with $\operatorname{rk} A = r < n$, then we have:

$D_1, D_2, \ldots, D_r > 0 \Rightarrow$ A positive semidefinite

$(-1)^i D_i > 0$ for $i = 1, 2, \ldots, r \Rightarrow$ A negative semidefinite

$(D_1 < 0, D_2 > 0, \ldots)$

c) $\text{Span}(\underline{v_1}, \underline{v_3}) = \underbrace{\text{all solutions of } A \cdot \underline{x} = \underline{0}}_{\text{Null}(A)}$

$A\underline{x} = \underline{0}$: $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 3 & 9 & -6 & 0 \\ 0 & -2 & -6 & 4 & 0 \end{array}\right) \xrightarrow[\text{from (a)}]{} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$$\begin{aligned} x &= 0 & & & x &= 0 \\ y &= -3z + 2w & &\Leftarrow & y + 3z - 2w &= 0 \end{aligned}$$

$\underline{z, w \text{: free}}$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -3z+2w \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -3z \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2w \\ 0 \\ w \end{pmatrix} = z \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + w \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Solutions $= \text{span}(\underline{v_1}, \underline{v_2})$. with $\underline{v_1} = \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

Revision:

$\text{Null}(A) = \text{span}(\underline{v_1}, \ldots, \underline{v_k})$ with $k = \#$ free variables and $\underline{v_1}, \ldots, \underline{v_k}$ lin. independent

$k = \dim \text{Null}(A)$

2.

a) $y' - 2y = e^t$

linear — int. factor / superposition

Superposition:

$$y = y_h + y_p = \underline{\underline{C \cdot e^{2t} - e^t}}$$

y_h: $y' - 2y = 0$

$r - 2 = 0$

$r = 2 \longrightarrow y_h = \underline{C \cdot e^{2t}}$

y_p: $y' - 2y = e^t$

$f = e^t$, $f' = e^t$

$\begin{cases} y = A \cdot e^t \\ y' = Ae^t \end{cases}$

$(Ae^t) - 2(Ae^t) = e^t$

$(A - 2A) \cdot e^t = e^t$

$(-A)\, e^t = 1 \cdot e^t$

$-A = 1$

$A = -1$

$y_p = -1 \cdot e^t = \underline{-e^t}$

Int. factor: $u = e^{\int -2\,dt} = e^{-2t + c} \Rightarrow \underline{u = e^{-2t}}$

$(y \cdot e^{-2t})' = e^t \cdot e^{-2t} = e^{-t}$

$y e^{-2t} = \int e^{-t}\,dt = -e^{-t} + C$

$y = e^{2t}(-e^{-t} + C) = \underline{\underline{-e^t + Ce^{2t}}}$

b) $3t^2 - y - ty' = 0$

linear

exact? Yes!

Exact: $\underbrace{(3t^2-y)}_{p} + \underbrace{(-t)}_{q} \cdot y' = 0$

$h'_t = 3t^2 - y$

$h'_y = -t$

$h = \int 3t^2 - y \, dt = t^3 - yt + g(y)$

$h'_y = -t$

$0 - \cancel{t} + g'(y) = \cancel{-t}$

$g'(y) = 0$

may choose $g(y) = 0$

Exact equation: $h(t,y) = t^3 - yt$

$t^3 - yt = C$

$\frac{-yt}{-t} = \frac{C - t^3}{-t}$

$y = \frac{C - t^3}{-t} = \frac{t^3 - C}{t} = t^2 - \frac{C}{t}$

Linear: $3t^2 - y - ty' = 0$

$-ty' - y = -3t^2 \quad |:(-t)$

$y' + \frac{1}{t} y = 3t$

Int. factor: $u = e^{\int 1/t \, dt} = e^{\ln|t| + C} = t$

$(y \cdot t)' = 3t^2$

$\frac{yt}{t} = \frac{\int 3t^2 dt}{t} = \frac{t^3 + C}{t}$

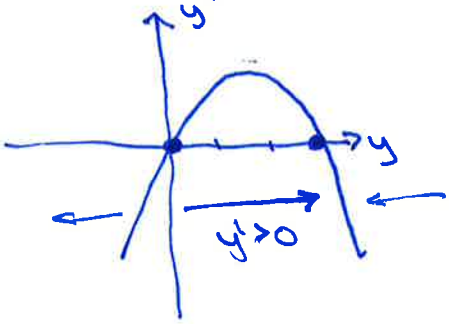
$y = t^2 + C/t$

c) $y' = 2y(3-y)$

Eq. states: $y' = 0$ $\quad 2y(3-y) = 0$

$\underline{\underline{y_e = 0}}$, $\underline{\underline{y_e = 3}}$

Stability:



phase diagram

$2y(3-y) = 6y - 2y^2$ $\cap$

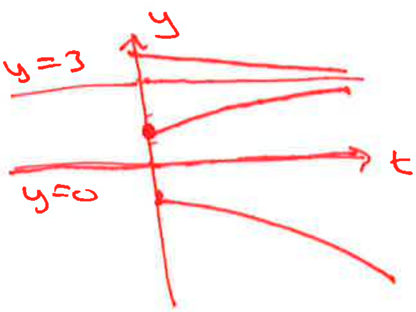
Answer:

$y_e = 3$ is <u>stable</u>

$y_e = 0$ is <u>unstable</u>

$y_e = 3$ is <u>not gl. as. stable</u>

($y_0 < 0$ means $y(t) \to -\infty$)



Stability Thm:

$F'(y_e) < 0 \Rightarrow y_e$ is stable

$F'(y_e) > 0 \Rightarrow y_e$ is unstable

$F' = 6 - 4y$

$F'(0) = 6$ $\quad y_e = 0$ unstable

$F'(3) = -6$ $\quad y_e = 3$ stable

5. $\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$

$\underline{y}' = A \cdot \underline{y}$

linear system of diff-eqn's.

Solution:

$\underline{y} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \cdot \underline{v_2} e^{\lambda_2 t}$

$\begin{vmatrix} 5-\lambda & -6 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda_1 = \underline{-1}, \lambda_2 = \underline{4}$

$\underline{\lambda = -1}$: $\begin{pmatrix} 6 & -6 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y - z = 0$ $\quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

z free

$\underline{\lambda = 4}$: $\begin{pmatrix} 1 & -6 \\ 1 & -6 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$y - 6z = 0$ $\quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 6z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

z free

(or y free $\quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y/6 \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1/6 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 1 \\ 1/6 \end{pmatrix}$)

General solution:

$$\begin{pmatrix} y \\ z \end{pmatrix} = C_1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot e^{-t} + C_2 \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix} e^{4t} = \underline{\underline{\begin{pmatrix} C_1 e^{-t} + 6 C_2 e^{4t} \\ C_1 e^{-t} + C_2 e^{4t} \end{pmatrix}}}$$

3. a) $u = 1 + x^2 + 5y^2 + 8z^2 + 4xy - 2yz$

$$u'_x = 2x + 4y$$
$$u'_y = 10y + 4x - 2z$$
$$u'_z = 16z - 2y$$

$$\Rightarrow H(u) = \begin{pmatrix} 2 & 4 & 0 \\ 4 & 10 & -2 \\ 0 & -2 & 16 \end{pmatrix}$$

$D_1 = 2$

$D_2 = 4$

$D_3 = 16 \cdot 4 + 2 \cdot \begin{vmatrix} 2 & 0 \\ 4 & -2 \end{vmatrix}$

$= 64 + 2(-4) > 0$

$H(u)$ pos. defn. for all x, y, z

$\Downarrow$

u convex

$\Leftarrow$

u convex means that any stationary pt. is a global min.

$(0,0,0)$ is stationary

$u_{min} = u(0,0,0) = 1$

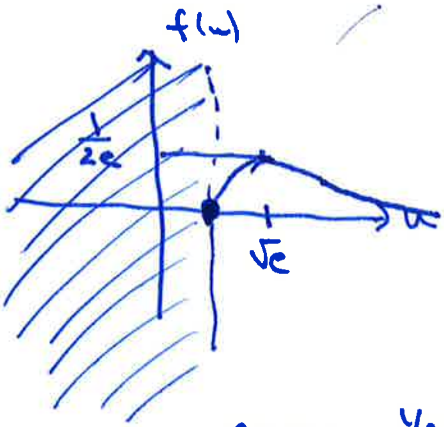
b) $f(u) = \dfrac{\ln(u)}{u^2}$, $u = u(x,y,z) = 1 + x^2 + \ldots$

$$f'_x = f'_u \cdot u'_x = \frac{1 - 2\ln u}{u^3} \cdot (2x + 4y)$$
$$f'_y = f'_u \cdot u'_y = \frac{1 - 2\ln u}{u^2} \cdot (4x + 10y - 2z)$$
$$f'_z = f'_u \cdot u'_z = \frac{1 - 2\ln u}{u^2} \cdot (-2y + 16z)$$

$$f'_u = \left(\frac{\ln u}{u^2}\right)'_u = \frac{\frac{1}{u} \cdot u^2 - \ln u \cdot 2u}{u^4} = \frac{u - 2u \cdot \ln u}{u^4}$$

$$= \frac{\cancel{u} \cdot (1 - 2\ln u)}{u^{\cancel{4}\,3}} = \frac{1 - 2\ln u}{u^3}$$

c) $f(u) = \dfrac{\ln u}{u^2}$, where $u = 1 + x^2 + \dots$



$\underline{u:}$ $u_{min} = 1$ from (a)

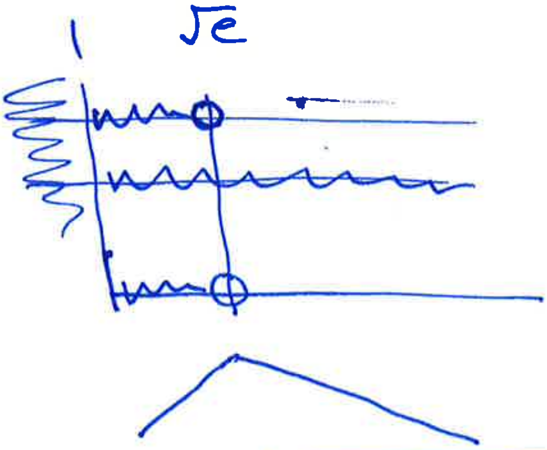
$\underline{f(u)} = \underline{\dfrac{\ln u}{u^2}}$, $u \geq 1$

$f(1) = \dfrac{\ln 1}{1^2} = \dfrac{0}{1} = 0$

$f'(u) = \dfrac{1 - 2\ln u}{u^3}$ from (b)

$f_{max} = f(\sqrt{e}) = \dfrac{1/2}{e} = \underline{\underline{\dfrac{1}{2e}}}$

$f_{min} = f(1) = \underline{\underline{0}}$



$1 - 2\ln u = 0$

$\ln u = 1/2$

$u = e^{1/2}$

$= \sqrt{e}$

$\lim\limits_{u\to\infty} \dfrac{\ln u}{u^2} = 0$ $\Bigg\{$ or $\lim\limits_{u\to\infty} \dfrac{\ln u}{u^2} = \lim\limits_{u\to\infty} \dfrac{1/u}{2u} = \lim\limits_{u\to\infty} \dfrac{1}{2u^2} = 0$

(L'Hopital's rule)

4. a) max $f = x^2y^2$ when $x^2+y^2+x^2y^2 \le 3$

KT problem in std. form.

KT conditions: FOC + C + CSC

$\mathcal{L} = x^2y^2 - \lambda\cdot(x^2+y^2+x^2y^2)$

FOC: $\mathcal{L}'_x = 2xy^2 - \lambda\cdot(2x+2xy^2) = 0$

$\mathcal{L}'_y = 2x^2y - \lambda\cdot(2y+2x^2y) = 0$

C: $x^2+y^2+x^2y^2 \le 3$

CSC: $\lambda \ge 0$ and $\lambda\cdot(x^2+y^2+x^2y^2-3) = 0$

b) Assume $x, y \ne 0$:

i) $x^2+y^2+x^2y^2 < 3$

$\lambda = 0 \Rightarrow 2xy^2 = 0$, $2x^2y = 0$

no solutions with $x,y \ne 0$

What if $\lambda = 1$?

$\lambda = 1 \Rightarrow \lambda = 0$ impossible

ii) $x^2+y^2+x^2y^2 = 3$: $\lambda \ge 0$

a) $2xy^2 - \lambda\cdot(2x+2xy^2) = 0$

$2x(y^2 - \lambda - \lambda y^2) = 0 \quad \leftarrow x \ne 0$

$y^2 - \lambda - \lambda y^2 = 0$

$y^2\cdot(1-\lambda) = \lambda \Rightarrow y^2 = \frac{\lambda}{1-\lambda}$

b) $2y(x^2 - \lambda - \lambda x^2) = 0 \quad \leftarrow y \ne 0$

$x^2 - \lambda - \lambda x^2 = 0$

$x^2\cdot(1-\lambda) = \lambda \Rightarrow x^2 = \frac{\lambda}{1-\lambda}$

a) b):

$\Rightarrow x^2 = y^2$: C:

$$x^2 + x^2 + x^2 \cdot x^2 = 3$$
$$x^4 + 2x^2 - 3 = 0$$

$v = x^2$: $v^2 + 2v - 3 = 0$

$$v = \frac{-2 \pm \sqrt{4 - 4\cdot(-3)}}{2} = \frac{-2 \pm 4}{2} = 1, -3$$

$x^2 = 1$ or $\cancel{x^2 = -3}$

$\underline{x = \pm 1}$

$y^2 = 1$

$\underline{y = \pm 1}$

$x^2 = \frac{\lambda}{1-\lambda}$ $\quad 1 = \frac{\lambda}{1-\lambda}$

$1 - \lambda = \lambda$

$1 = 2\lambda$

$\underline{\lambda = 1/2}$

Four ~~Four~~ candidate pts with $x, y \neq 0$:

$(x, y; \lambda) = \underline{(\pm 1, \pm 1; 1/2)}$ $\quad \boxed{f = 1}$ at all four pts since $f(x,y) = x^2 y^2$

c) No time for this today, will do it next week.