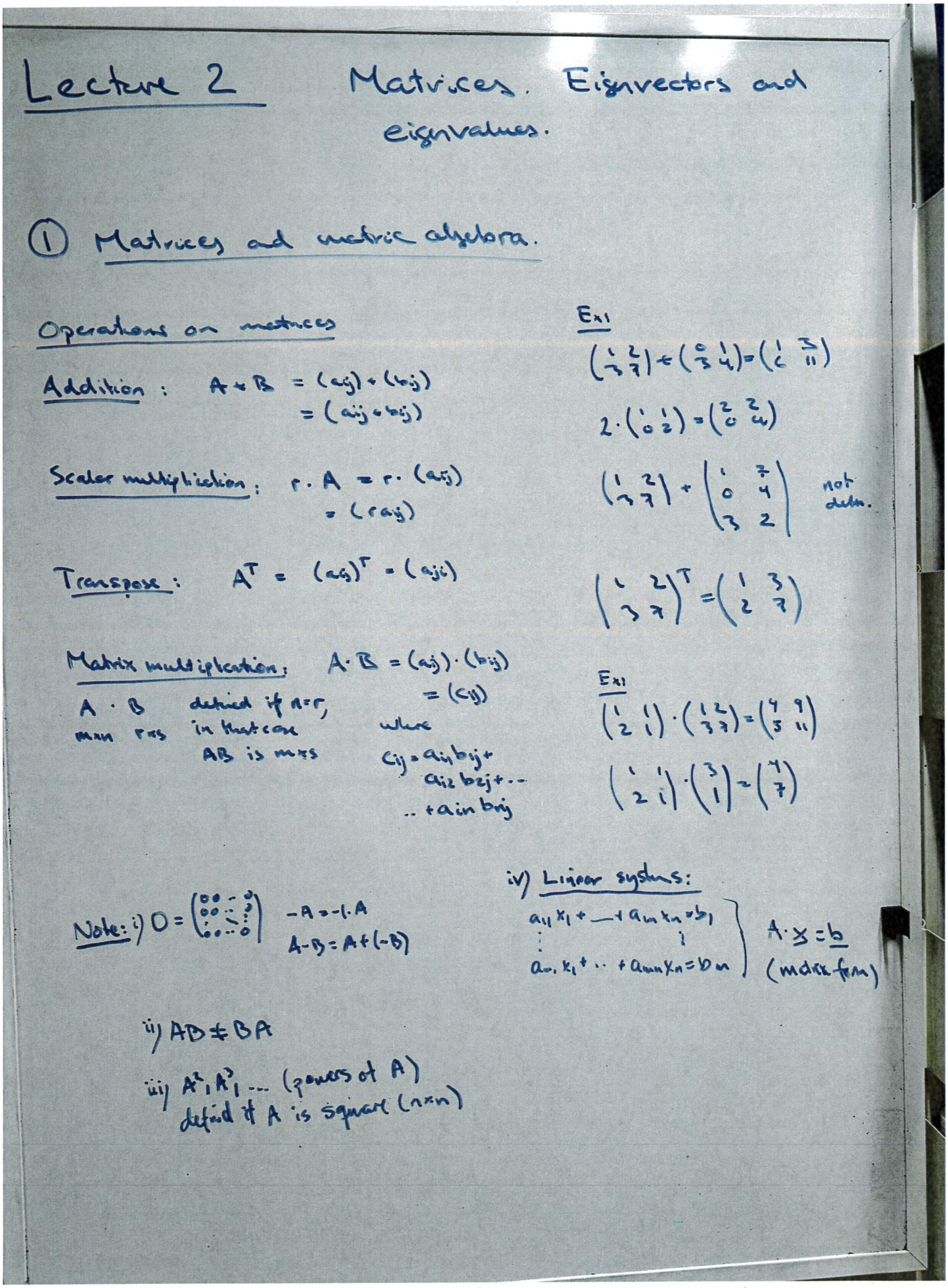


Lecture 2 Matrices. Eigenvectors and eigenvalues.

① Matrices and matric algebra.

Operations on matrices

Addition : $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

Scaler multiplication : $r \cdot A = r \cdot (a_{ij}) = (r a_{ij})$

Transpose : $A^T = (a_{ij})^T = (a_{ji})$

Matrix multiplcation : $A \cdot B = (a_{ij}) \cdot (b_{ij}) = (c_{ij})$

$A \cdot B$ defined if $n = r$,
$m \times n$ $r \times s$ in that case
AB is $m \times s$

where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$

Ex1

$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 6 & 11 \end{pmatrix}$$

$$2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 7 \\ 0 & 4 \\ 3 & 2 \end{pmatrix}$$ not defn.

$$\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

Ex1

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 9 \\ 5 & 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

Note: i) $0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ $-A = -1 \cdot A$, $A - B = A + (-B)$

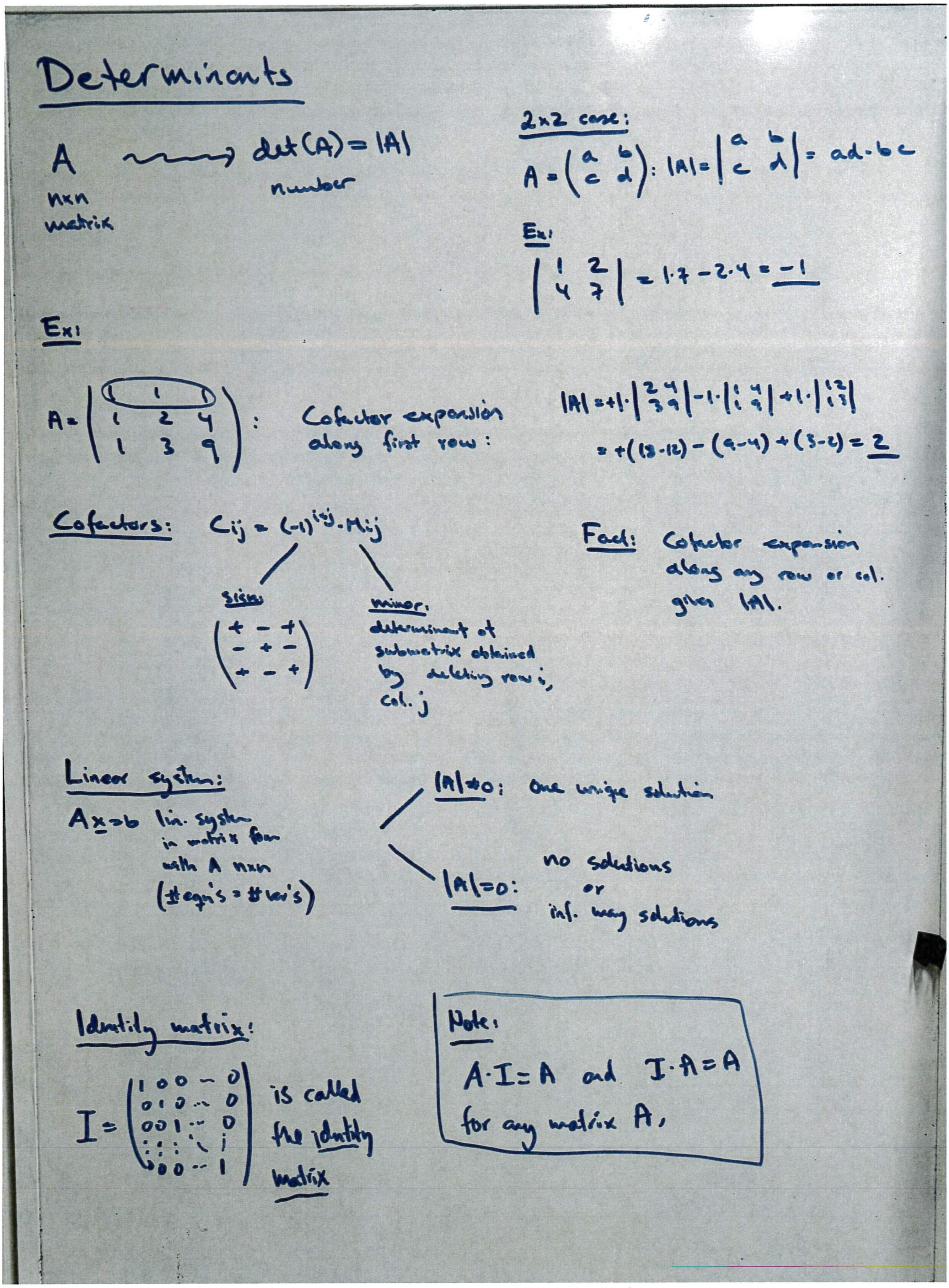
ii) $AB \neq BA$

iii) $A^2, A^3, \ldots$ (powers of A)
defind if A is square ($n \times n$)

iv) **Linear systems:**

$$\left.\begin{matrix} a_{11} x_1 + \ldots + a_{1n} x_n = b_1 \\ \vdots \qquad\qquad \vdots \\ a_{m1} x_1 + \ldots + a_{mn} x_n = b_m \end{matrix}\right\} \quad A \cdot \underline{x} = \underline{b}$$

(matrix form)



Determinants

$A \rightsquigarrow \det(A) = |A|$

$n \times n$ matrix

number

2x2 case:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex:

$\begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} = 1 \cdot 7 - 2 \cdot 4 = \underline{-1}$

Ex:

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$: Cofactor expansion along first row:

$|A| = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= +(18-12) - (9-4) + (3-2) = \underline{2}$

Cofactors: $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

sign:

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

minor: determinant of submatrix obtained by deleting row i, col. j

Fact: Cofactor expansion along any row or col. gives $|A|$.

Linear system:

$A\underline{x} = b$ lin. system in matrix form with A $n \times n$ (# eqn's = # var's)

$\underline{|A| \neq 0}$: One unique solution

$\underline{|A| = 0}$: no solutions or inf. many solutions

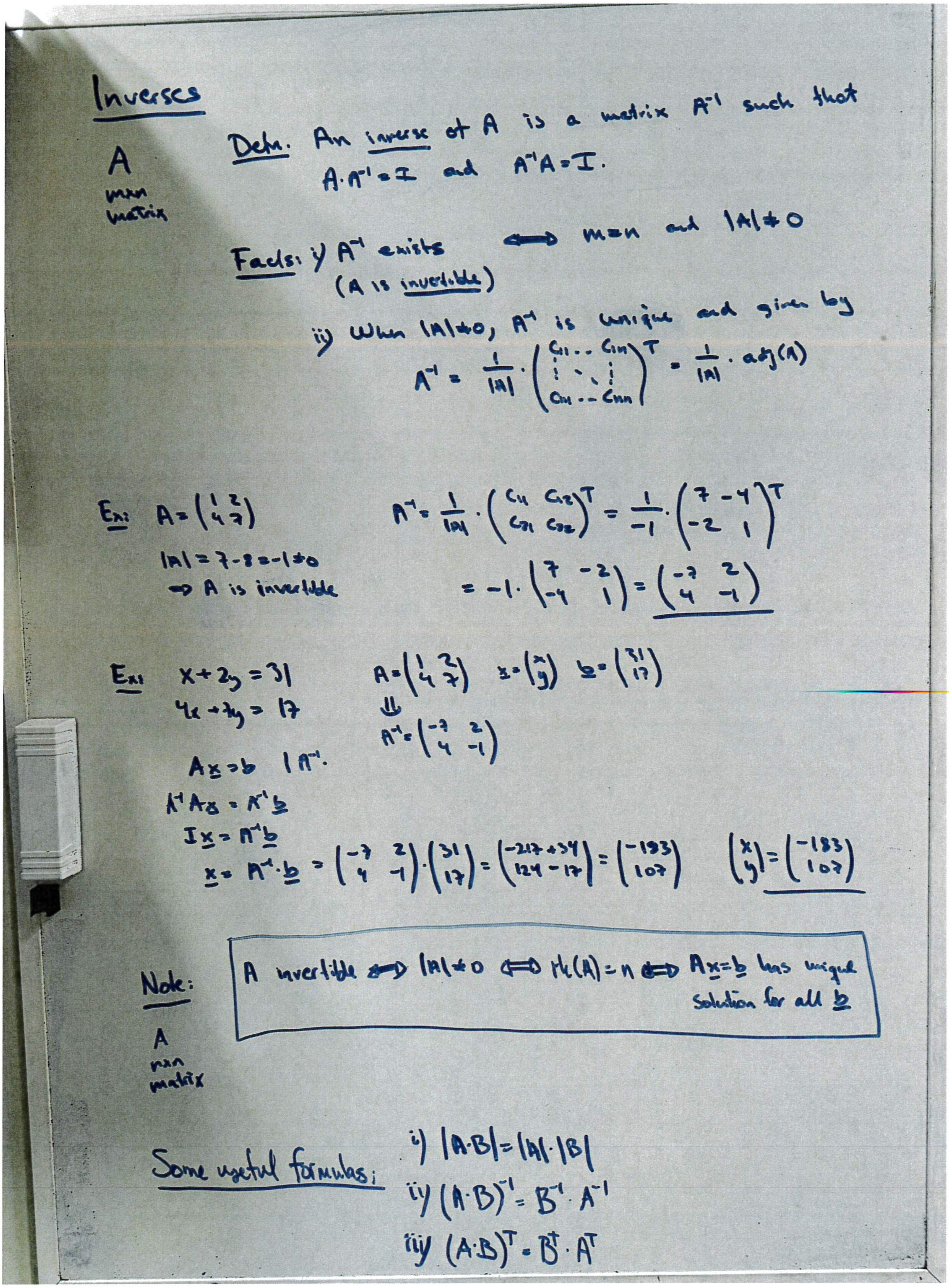
Identity matrix:

$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$ is called the identity matrix

Note:

$A \cdot I = A$ and $I \cdot A = A$

for any matrix A,



Inverses

A
$m\times n$
matrix

Defn. An *inverse* of A is a matrix A^{-1} such that
$A\cdot A^{-1} = I$ and $A^{-1}A = I$.

Facts: i) A^{-1} exists $\Longleftrightarrow$ $m=n$ and $|A|\neq 0$
(A is *invertible*)

ii) When $|A|\neq 0$, A^{-1} is unique and given by

$$A^{-1} = \frac{1}{|A|}\cdot\begin{pmatrix} C_{11} & \cdots & C_{1n}\\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn}\end{pmatrix}^T = \frac{1}{|A|}\cdot \operatorname{adj}(A)$$

Ex: $A=\begin{pmatrix}1&2\\4&7\end{pmatrix}$

$|A| = 7-8 = -1 \neq 0$
$\Rightarrow$ A is invertible

$$A^{-1} = \frac{1}{|A|}\cdot\begin{pmatrix}C_{11}&C_{12}\\C_{21}&C_{22}\end{pmatrix}^T = \frac{1}{-1}\cdot\begin{pmatrix}7&-4\\-2&1\end{pmatrix}^T$$

$$= -1\cdot\begin{pmatrix}7&-2\\-4&1\end{pmatrix} = \underline{\begin{pmatrix}-7&2\\4&-1\end{pmatrix}}$$

Ex: $x+2y=31$
$4x+7y=17$

$A=\begin{pmatrix}1&2\\4&7\end{pmatrix}$ $\underline{x}=\begin{pmatrix}x\\y\end{pmatrix}$ $\underline{b}=\begin{pmatrix}31\\17\end{pmatrix}$
$\Downarrow$
$A^{-1}=\begin{pmatrix}-7&2\\4&-1\end{pmatrix}$

$A\underline{x}=\underline{b} \quad | A^{-1}\cdot$
$A^{-1}A\underline{x} = A^{-1}\underline{b}$
$I\underline{x} = A^{-1}\underline{b}$
$\underline{x} = A^{-1}\cdot\underline{b} = \begin{pmatrix}-7&2\\4&-1\end{pmatrix}\cdot\begin{pmatrix}31\\17\end{pmatrix} = \begin{pmatrix}-217+34\\124-17\end{pmatrix} = \begin{pmatrix}-183\\107\end{pmatrix}$ $\quad \underline{\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-183\\107\end{pmatrix}}$

Note:

A
$n\times n$
matrix

$$\boxed{A \text{ invertible} \Longleftrightarrow |A|\neq 0 \Longleftrightarrow \operatorname{rk}(A)=n \Longleftrightarrow A\underline{x}=\underline{b} \text{ has unique solution for all } \underline{b}}$$

Some useful formulas:
i) $|A\cdot B| = |A|\cdot|B|$
ii) $(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}$
iii) $(A\cdot B)^T = B^T\cdot A^T$

Defn.

A $n \times n$ matrix

A is called <u>diagonal</u> if $a_{ij}=0$ for all $i \neq j$

— ‖ — <u>upper triangular</u> if $a_{ij}=0$ for $i>j$.

Ex: $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is <u>diagonal</u>

$A=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 7 \end{pmatrix}$ is <u>upper triangular</u>

A is called <u>ortogonal</u> if A is invertible and $A^{-1}=A^T$

Note: $A=\left(\underline{v_1}\middle|\underline{v_2}\middle|\dots\middle|\underline{v_n}\right)$ is orthogonal $\Longleftrightarrow \begin{cases} \text{i) } \underline{v_i}\cdot\underline{v_j}=0 & \text{for } i \neq j \\ \text{ii) } \|v_i\|=1 & \text{for all } i \end{cases}$

If these conditions hold, $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ is called an <u>orthonormal base</u> of $\mathbb{R}^n$

Ex: $A=\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ is orthogonal since $\begin{cases} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -2+2=0 \\ \left\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\| = \sqrt{4+1}=\sqrt{5} \\ \left\|\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\| = \sqrt{1+4}=\sqrt{5} \end{cases}$

A $m \times n$ matrix

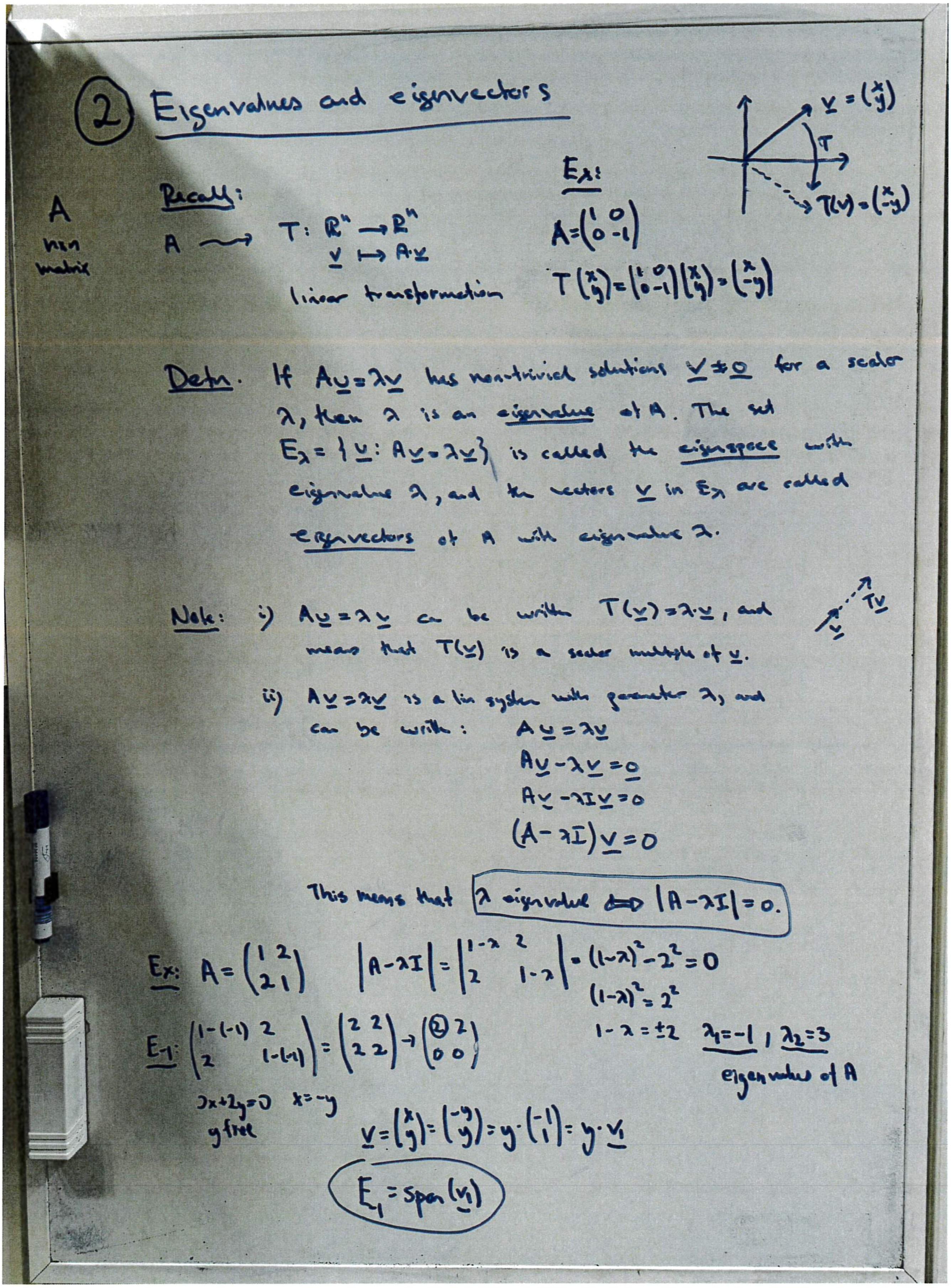
Defn A is symmetric if $A^T=A$.

Fact: A symmetric $\Longleftrightarrow$ $m=n$ and $a_{ij}=a_{ji}$

Ex: $A=\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

is symmetric

$(a_{12}=a_{21}=2)$



② Eigenvalues and eignvectors

A
n×n
matrix

Recall:

$A \rightsquigarrow T: \mathbb{R}^n \to \mathbb{R}^n$

$\underline{v} \mapsto A\cdot\underline{v}$

linear transformation

Ex:

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

Defn. If $A\underline{v} = \lambda\underline{v}$ has non-trivial solutions $\underline{v} \neq \underline{0}$ for a scalar λ, then λ is an eigenvalue of A. The set $E_\lambda = \{\underline{v} : A\underline{v} = \lambda\underline{v}\}$ is called the eigenspace with eigenvalue λ, and the vectors $\underline{v}$ in E_λ are called eigenvectors of A with eigenvalue λ.

Note: i) $A\underline{v} = \lambda\underline{v}$ can be written $T(\underline{v}) = \lambda\cdot\underline{v}$, and means that $T(\underline{v})$ is a scalar multiple of $\underline{v}$.

ii) $A\underline{v} = \lambda\underline{v}$ is a lin system with parameter λ, and can be written:

$$A\underline{v} = \lambda\underline{v}$$
$$A\underline{v} - \lambda\underline{v} = \underline{0}$$
$$A\underline{v} - \lambda I\underline{v} = 0$$
$$(A - \lambda I)\underline{v} = 0$$

This means that $\boxed{\lambda \text{ eigenvalue} \Leftrightarrow |A - \lambda I| = 0.}$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2 = 0$

$(1-\lambda)^2 = 2^2$

$1 - \lambda = \pm 2 \quad \underline{\lambda_1 = -1}, \ \underline{\lambda_2 = 3}$

eigenvalues of A

E_{-1}: $\begin{pmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} ② & 2 \\ 0 & 0 \end{pmatrix}$

$2x + 2y = 0 \quad x = -y$

y free

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix} = y\cdot\underline{v_1}$

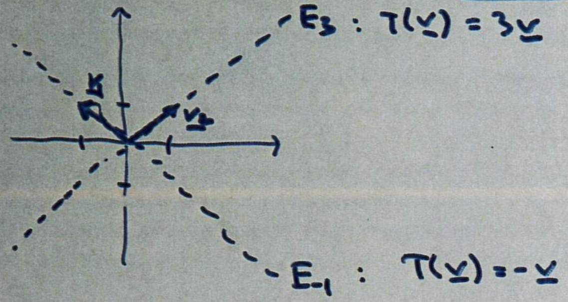
$E_{-1} = \text{Span}(\underline{v_1})$

$\underline{E_3}$: $\begin{pmatrix} 1-3 & 2 \\ 2 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} \textcircled{-2} & 2 \\ 0 & 0 \end{pmatrix}$

$-2x+2y=0 \quad y=x$
y free

$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = y \cdot \underline{v_2}$

$$\boxed{E_3 = \operatorname{span}(\underline{v_2})}$$



Note:

$\{\underline{v_1}, \underline{v_2}\}$ is a base of $\mathbb{R}^2$

$\Downarrow$

Any vector $\underline{v}$ can be written uniquely as:

$$\underline{v} = c_1 \cdot \underline{v_1} + c_2 \cdot \underline{v_2}$$

$$\Rightarrow T(\underline{v}) = c_1 T(\underline{v_1}) + c_2 T(\underline{v_2})$$
$$= c_1 \cdot (-\underline{v_1}) + c_2 (3\underline{v_2})$$
$$= -c_1 \underline{v_1} + 3c_2 \underline{v_2}$$

In general:

A
$n \times n$ matrix

i) The characteristic equation $|A - \lambda I| = 0$ is a polynomial eqn. of degree n. There are maximally n eigenvalues of A.

ii) For any eigenvalue λ, $E_\lambda = \{\underline{v} : (A - \lambda I)\underline{v} = \underline{0}\} = \operatorname{Null}(A - \lambda I)$. It is a linear subspace of $\mathbb{R}^n$.

iii) Let m_{λ_i} be the algebraic multiplicity of an eigenvalue λ_i, i.e. the maximal m s.t. $(\lambda - \lambda_i)^m$ is a factor of $|A - \lambda I| = 0$. Then $\boxed{1 \leq \dim E_\lambda \leq m_\lambda}$ for all eigenvalues λ.

Defn: A is diagonalizable if there is a diagonal matrix D and an invertible matrix P s.t. $P^{-1}AP = D$. It is orthogonally diagonalizable if we may choose P as an orthogonal matrix, i.e. s.t. $P^{-1} = P^T$.

Note: If $P = (\underline{v_1} | \cdots | \underline{v_n})$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ then we have:

$AP = (A\underline{v_1} | \cdots | A\underline{v_n})$ and $PD = (\lambda_1 \underline{v_1} | \cdots | \lambda_n \underline{v_n})$

This means that $AP=PD$ if and only if $A\underline{v_i}=\lambda_i\underline{v_i}$ for all i,
and $AP=PD \iff P^{-1}AP=D$ if and only if P is invertible.

__Result:__ The following are equivalent:

A
n×n
matrix

i) A is diagonalizable
ii) A has n eigenvalue (counted with multiplicity) and n linearly independent eigenvectors.
iii) A has n eigenvalues (counted with multiplicity) and $\dim E_\lambda = m_\lambda$ for each eigenvalue with $m_\lambda > 1$.

__Ex:__ i) $A=\begin{pmatrix}1&2\\2&1\end{pmatrix}$ $\quad \lambda_1=-1 \quad \underline{v_1}=\begin{pmatrix}-1\\1\end{pmatrix}$, $\quad \lambda_2=3 \quad \underline{v_2}=\begin{pmatrix}1\\1\end{pmatrix}$ $\quad$ A __diagonalizable__ $\begin{cases} D=\begin{pmatrix}-1&0\\0&3\end{pmatrix} \\ P=\begin{pmatrix}-1&1\\1&1\end{pmatrix}\end{cases}$

ii) $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $\quad \begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=\lambda^2+1=0$ $\quad$ A __not diag.__

__no real eigenvalues__

iii) $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\quad \begin{vmatrix}1-\lambda&1\\0&1-\lambda\end{vmatrix}=(1-\lambda)^2=0$ $\quad D=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\underline{\lambda_1=\lambda_2=1}$ or $\underline{\lambda=1}$ with $m_\lambda=2$ $\quad P=\begin{pmatrix}1&?\\0&?\end{pmatrix}$

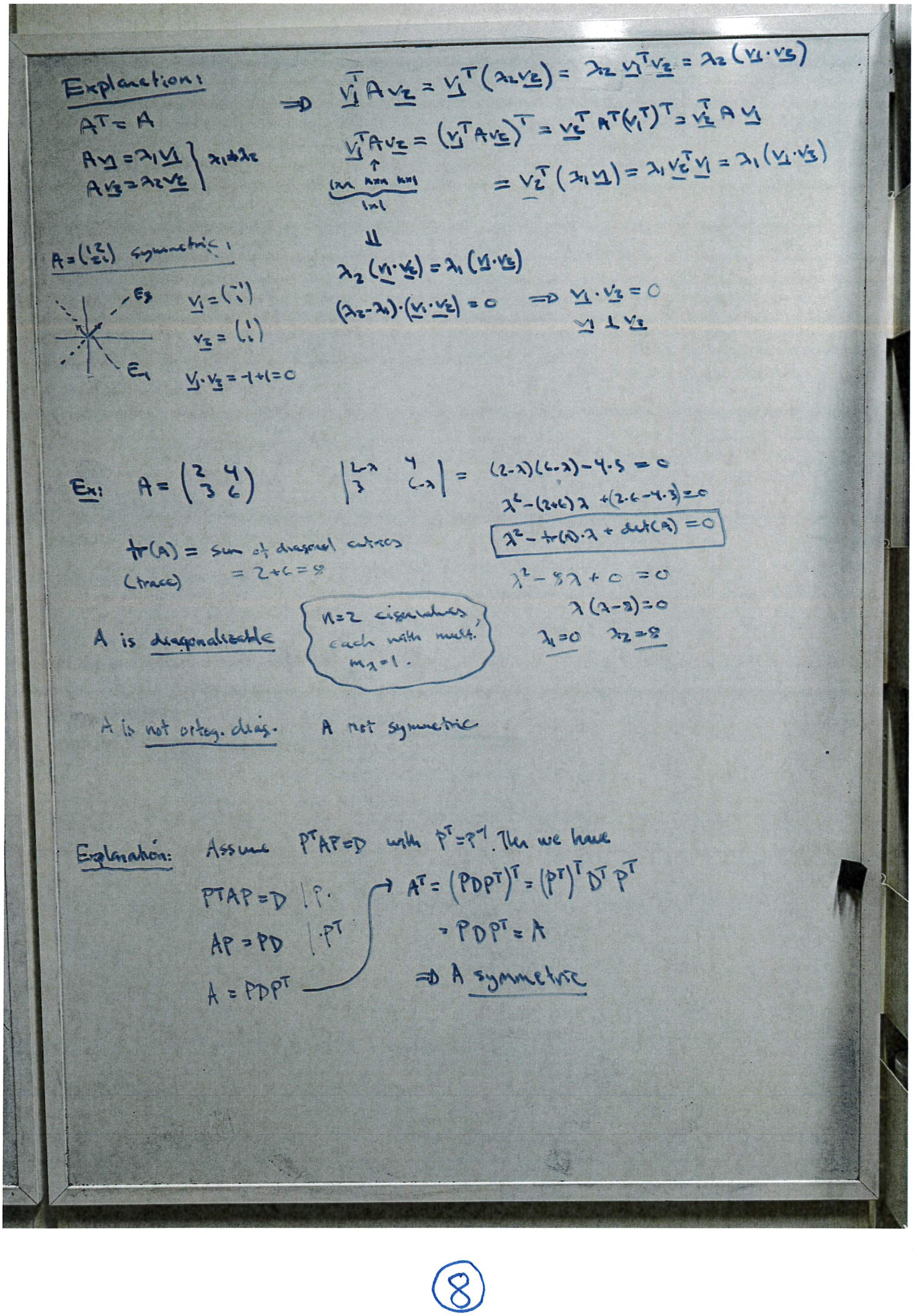
$\underline{E_1}$: $\begin{pmatrix}0&\textcircled{1}\\0&0\end{pmatrix}$ $\quad$ $y=0$, x free $\quad \underline{v}=\begin{pmatrix}x\\0\end{pmatrix}=x\cdot\begin{pmatrix}1\\0\end{pmatrix}=\operatorname{span}\begin{pmatrix}1\\0\end{pmatrix}$ $\quad$ A __not diag__

$\dim E_1 = 1 < m_1 = 2$

__Note:__ i) If $\underline{v_1}\in E_{\lambda_1}$ and $\underline{v_2}\in E_{\lambda_2}$ with $\lambda_1 \neq \lambda_2$, then $\{\underline{v_1},\underline{v_2}\}$ are __lin. independent__
ii) — " — and A is symmetric, then $\underline{v_1} \perp \underline{v_2}$.

__Result:__ $\boxed{A \text{ ortogonally diagonalizable} \iff A \text{ symmetric}}$

A
n×n
matrix



<u>**Explanation:**</u>

$A^T = A$

$$\left.\begin{array}{l} A\underline{v_1} = \lambda_1 \underline{v_1} \\ A\underline{v_2} = \lambda_2 \underline{v_2} \end{array}\right\} \lambda_1 \neq \lambda_2$$

$$\Rightarrow \underline{v_1}^T A \underline{v_2} = \underline{v_1}^T(\lambda_2 \underline{v_2}) = \lambda_2 \underline{v_1}^T \underline{v_2} = \lambda_2 (\underline{v_1} \cdot \underline{v_2})$$

$$\underset{\underbrace{1\times n \quad n\times n \quad n\times 1}_{1\times 1}}{\underline{v_1}^T A \underline{v_2}} = (\underline{v_1}^T A \underline{v_2})^T = \underline{v_2}^T A^T (\underline{v_1}^T)^T = \underline{v_2}^T A \underline{v_1}$$

$$= \underline{v_2}^T(\lambda_1 \underline{v_1}) = \lambda_1 \underline{v_2}^T \underline{v_1} = \lambda_1 (\underline{v_1} \cdot \underline{v_2})$$

$$\Downarrow$$

$$\lambda_2 (\underline{v_1} \cdot \underline{v_2}) = \lambda_1 (\underline{v_1} \cdot \underline{v_2})$$

$$(\lambda_2 - \lambda_1) \cdot (\underline{v_1} \cdot \underline{v_2}) = 0 \Rightarrow \underline{v_1} \cdot \underline{v_2} = 0$$

$$\underline{v_1} \perp \underline{v_2}$$

<u>$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ symmetric:</u>

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{v_1} \cdot \underline{v_2} = -1 + 1 = 0$

<u>Ex:</u> $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{vmatrix} = (2-\lambda)(6-\lambda) - 4\cdot 3 = 0$$

$$\lambda^2 - (2+6)\lambda + (2\cdot 6 - 4\cdot 3) = 0$$

$$\boxed{\lambda^2 - tr(A)\cdot\lambda + det(A) = 0}$$

$tr(A)$ = sum of diagonal entries

(trace) $= 2 + 6 = 8$

$$\lambda^2 - 8\lambda + 0 = 0$$

$$\lambda(\lambda - 8) = 0$$

$$\underline{\lambda_1 = 0} \quad \underline{\lambda_2 = 8}$$

A is <u>diagonalizable</u>

$n = 2$ eigenvalues, each with mult. $m_\lambda = 1$.

A is <u>not orthog. diag.</u> A not symmetric

<u>Explanation:</u> Assume $P^TAP = D$ with $P^T = P^{-1}$. Then we have

$P^TAP = D \quad | P\cdot$

$AP = PD \quad | \cdot P^T$

$A = PDP^T \longrightarrow A^T = (PDP^T)^T = (P^T)^T D^T P^T$

$= PDP^T = A$

$\Rightarrow$ A <u>symmetric</u>

Orthonormal bases

Ex: $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

symmetric 3×3 - matrix

$\Rightarrow$ there is an orthogonal matrix $P = (\underline{v}_1 | \underline{v}_2 | \underline{v}_3)$

s.t. $P^TAP = D$ is diagonal

Char. eqn:

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left((1-\lambda)^2-4\right) - 2\left(2(1-\lambda)-4\right) + 2\left(4-2(1-\lambda)\right) = 0$$

$$(1-\lambda)(\lambda^2-2\lambda-3) - 2(-2\lambda-2) + 2(2+2\lambda) = 0$$

$$(1-\lambda)(\lambda-3)(\lambda+1) + 4(1+\lambda) = 0$$

$$(\lambda+1)\left((1-\lambda)(\lambda-3)+4\right) = 0$$

$\underline{\lambda=-1}$, $-\lambda^2+4\lambda+1=0$

$\lambda^2-4\lambda-1=0$

$\underline{\lambda=5}$, $\lambda\underline{=-1}$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

E_{-1} E_5

$\underline{E_5}$: $\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $-2x+y+z=0 \Rightarrow 2x=2z \Rightarrow \underline{x=z}$

$-3y+3z=0 \Rightarrow \underline{y=z}$

$\underline{z \text{ free}}$

$\underline{v} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{3}$

$\underline{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ← orthonormal base of E_5

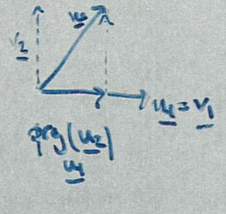
$\underline{E_{-1}}$: $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} ① & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x = -y-z$, y,z free $\Big\}$ $\underline{v} = \begin{pmatrix} -y-z \\ y \\ z \end{pmatrix}$

$= y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \underline{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ ← base of E_{-1}

$\underline{u}_1 \perp \underline{v}_3$, $\underline{u}_2 \perp \underline{v}_3$, but $\underline{u}_1 \cdot \underline{u}_2 = 1 \neq 0$

$\underline{v}_1 = \underline{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \underline{v}_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$



$\underline{v}_3 = \underline{u}_2 - \text{proj}_{\underline{u}_1}(\underline{u}_2)$

$= \underline{u}_2 - \frac{\underline{u}_2 \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\underline{v}_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \Rightarrow \underline{v}_3' = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$\text{span}(\underline{v}_1', \underline{v}_2') = \text{span}(\underline{u}_1, \underline{u}_2) = E_{-1}$

$$P = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$$

↑ E_{-1} ↑ E_5

$\{\underline{v}_1', \underline{v}_2'\}$ orthonormal base of E_{-1}

In general:

Gramm-Schmidt process to find an orthonormal base.