Lecture 1: Vectors, vector spaces and linear systems.

Notation:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (v_i)$$

n-vector

(v_i: i'th component)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

$m \times n$-matrix

(a_{ij}: element in position (i,j))

① Vectors

$\mathbb{R}^n$ = Euclidean n-space = set of all n-vectors

Addition: $\underline{v} + \underline{w} = (v_i) + (w_i) = (v_i + w_i)$

Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

Scalar multiplication: $r \cdot \underline{v} = r(v_i) = (r v_i)$

$2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

Note: $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$-\underline{v} = -(v_{ij}) = (-v_{ij})$

$\underline{v} - \underline{w} = \underline{v} + (-\underline{w})$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

Norm (length): $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

Ex: $\left\| \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

Inner product (dot product):

$\underline{v} \cdot \underline{w} = \langle \underline{v}, \underline{w} \rangle = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

$\underline{v} \cdot \underline{v} = \|\underline{v}\|^2 \geq 0$

Ex: $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 3 - 2 = 1$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 + 4 = 5$

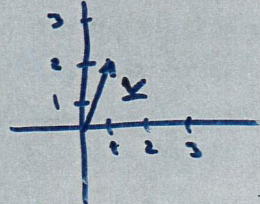
Triangle inequality: $\|\underline{v}+\underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$

Cauchy-Schwarz' inequality: $|\underline{v}\cdot\underline{w}| \leq \|\underline{v}\|\cdot\|\underline{w}\|$

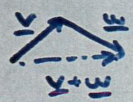
Geometric interpretation of vectors

- Vector $\underline{v}$ given by its **length** and **direction**, i.e. a **translation** represented as an arrow

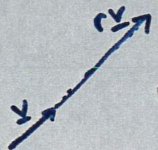
Ex: $\underline{v} = \begin{pmatrix}1\\2\end{pmatrix} \rightsquigarrow$ Represented by:



Addition:



Scalar multiplication:

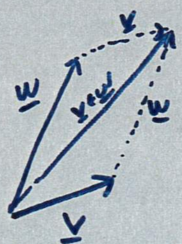


$r\underline{v}$ has the same direction as $\underline{v}$ and is scaled up with factor r, (or the opposite direction and scaled up with factor $|r|$ if $r<0$)

Note: Parallell arrows of the same length represent the same vector

Explanation: Triangle inequality

"diagonal in parallellogram is at least as long as the sum of the sides"



Pf: $(\|\underline{v}\|+\|\underline{w}\|)^2 = \|\underline{v}\|^2 + 2\|\underline{v}\|\cdot\|\underline{w}\| + \|\underline{w}\|^2$

$\|\underline{v}+\underline{w}\|^2 = (\underline{v}+\underline{w})\cdot(\underline{v}+\underline{w}) = \underline{v}\cdot\underline{v} + \underline{w}\cdot\underline{v} + \underline{v}\cdot\underline{w} + \underline{w}\cdot\underline{w}$ (bilinear inner product)

$= \|\underline{v}\|^2 + 2\,\underline{v}\cdot\underline{w} + \|\underline{w}\|^2$

Difference $= 2\|\underline{v}\|\cdot\|\underline{w}\| - 2\,\underline{v}\cdot\underline{w}$

$= 2(\|\underline{v}\|\cdot\|\underline{w}\| - \underline{v}\cdot\underline{w}) \geq 0$ by C.S. inequality ∎

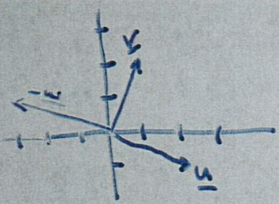
Orthogonality

$\underline{v} \perp \underline{w}$ ($\underline{v}$ and $\underline{w}$ orthogonal) if $\underline{v} \cdot \underline{w} = 0$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\underline{v} \cdot \underline{w} = 3 - 2 = 1$

$\underline{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $-\underline{w} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\underline{u} \cdot \underline{v} = 2 - 2 = 0$ $\underline{u} \perp \underline{v}$

Interpretation:

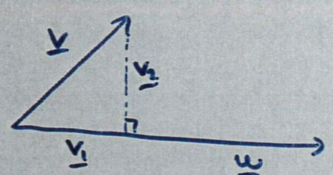


$\underline{u} \perp \underline{v}$: angle is 90°

$\underline{v} \cdot \underline{w} = 1 > 0$: angle is acute $(< 90^\circ)$

$-\underline{w} \cdot \underline{v} = -1 < 0$: angle is obtuse $(> 90^\circ)$

Orthogonal projection:



If $\underline{v}, \underline{w} \neq \underline{0}$ then there is a decomposition

$\underline{v} = \underline{v_1} + \underline{v_2}$ s.t. (1) $\underline{v_1}$ is along $\underline{w}$

(2) $\underline{v_2} \perp \underline{w}$

Then $\underline{v_1} = \text{proj}_{\underline{w}}(\underline{v})$ is called the orthogonal projection on $\underline{w}$. We have:

$$\text{proj}_{\underline{w}}(\underline{v}) = \left(\frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}} \right) \underline{w}$$

Explanation:

(1) means $\underline{v_1} = a\underline{w}$

(2) means $\underline{v_2} \cdot \underline{w} = 0$

$\Downarrow$

$\underline{v_2} \cdot \underline{w} = (\underline{v} - \underline{v_1}) \cdot \underline{w} = (\underline{v} - a\underline{w}) \cdot \underline{w} = 0$

$\underline{v} \cdot \underline{w} - a \cdot \underline{w} \cdot \underline{w} = 0$

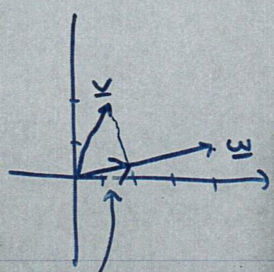
$$a = \frac{\underline{v} \cdot \underline{w}}{\underline{w} \cdot \underline{w}}$$

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\underline{v} \cdot \underline{w} = 6$

$\underline{w} \cdot \underline{w} = 17$

$\text{proj}_{\underline{w}}(\underline{v}) = \frac{6}{17} \underline{w}$



Explanation C-S inequality $|\underline{v}\cdot\underline{w}| \leq \|\underline{v}\|\cdot\|\underline{w}\|$

a) Case $\|\underline{v}\| = \|\underline{w}\| = 1$:

Unit circle

$$\text{proj}_{\underline{w}}(\underline{v}) = \frac{\underline{v}\cdot\underline{w}}{1}\cdot\underline{w} = (\underline{v}\cdot\underline{w})\cdot\underline{w}$$

We must have

$$-1 \leq \underline{v}\cdot\underline{w} \leq 1$$

$$\Rightarrow |\underline{v}\cdot\underline{w}| \leq 1 \quad \square$$

b) General case:

$$\underline{v}' = \frac{1}{\|\underline{v}\|}\underline{v} \qquad \underline{w}' = \frac{1}{\|\underline{w}\|}\underline{w}$$

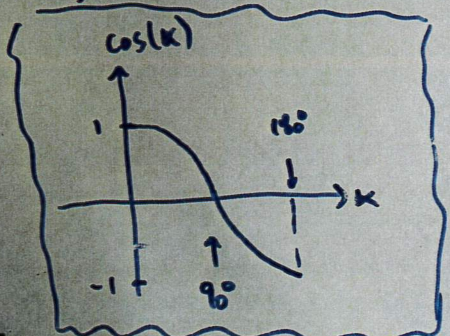
$$\Rightarrow |\underline{v}'\cdot\underline{w}'| \leq 1$$

$$\Rightarrow \|\underline{v}\|\cdot\|\underline{w}\|\cdot|\underline{v}'\cdot\underline{w}'| \leq \|\underline{v}\|\cdot\|\underline{w}\|$$

$$|\underline{v}\cdot\underline{w}| \leq \|\underline{v}\|\cdot\|\underline{w}\| \quad \square$$

The expression $\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|}$ is a measure of the angle α between vectors $\underline{v}$ and $\underline{w}$.

By C.S. inequality $-1 \leq \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} \leq 1$. In fact, $\frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\cdot\|\underline{w}\|} = \cos(\alpha)$



$\alpha = 0$: same direction $\cos(\alpha) = 1$

$\alpha = 180°$: opposite direction $\cos(\alpha) = -1$

Proof of C.S. inequality: $|\underline{v}\cdot\underline{w}| \leq \|\underline{v}\|\cdot\|\underline{w}\|$

If $\underline{w} = \underline{0}$: ok. Assume $\underline{w} \neq \underline{0}$. Put $\underline{u} = \underline{v} - \text{proj}_{\underline{w}}(\underline{v}) = \underline{v} - r\underline{w}$ $\quad r = \frac{\underline{v}\cdot\underline{w}}{\underline{w}\cdot\underline{w}}$

$$\Rightarrow \underline{v} = \underline{u} + r\underline{w}$$

$$\underline{v}\cdot\underline{v} = (\underline{u}+r\underline{w})\cdot(\underline{u}+r\underline{w}) = \underline{u}\cdot\underline{u} + 2r\,\underline{u}\cdot\underline{w} + r^2\,\underline{w}\cdot\underline{w} \quad \underline{u}\perp\underline{w}$$

$$= \|\underline{u}\|^2 + r^2\|\underline{w}\|^2 \geq r^2\|\underline{w}\|^2$$

$$\|\underline{v}\|^2 \geq \left(\frac{\underline{v}\cdot\underline{w}}{\underline{w}\cdot\underline{w}}\right)^2\cdot\|\underline{w}\|^2 = \frac{(\underline{v}\cdot\underline{w})^2}{\|\underline{w}\|^2} \Rightarrow \|\underline{v}\|^2\cdot\|\underline{w}\|^2 \geq (\underline{v}\cdot\underline{w})^2$$

$$\Rightarrow \|v\|\cdot\|w\| > |\underline{v}\cdot\underline{w}| \quad \square$$

Linear combinations

$\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ n-vectors

Defn A linear combination of $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_r}$ is a vector of the form

$$c_1 \underline{v_1} + c_2 \underline{v_2} + \ldots + c_r \underline{v_r}$$

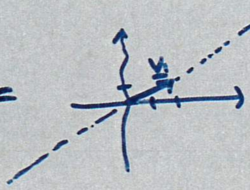
($c_1, \ldots, c_r$ any numbers)

$$\text{span}(\underline{v_1}, \ldots, \underline{v_r}) = \text{all lin. comb. of } \underline{v_1}, \ldots, \underline{v_r}$$
$$= \{ c_1 \underline{v_1} + c_2 \underline{v_2} + \ldots + c_r \underline{v_r} \mid c_1 \ldots c_r \text{ scalars} \}$$
$$\subseteq \mathbb{R}^n \quad (\text{subset of } \mathbb{R}^n)$$

Ex: $\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\text{span}(\underline{v_1}) = \{ c \cdot \underline{v_1} \mid c \text{ scalar} \}$

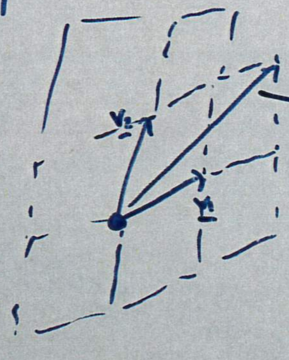
span($\underline{v_1}$) = line

= 

in $\mathbb{R}^2$ (two-dim coord. system)

$\underline{v_1} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

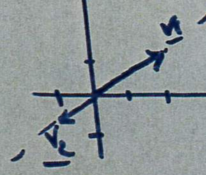
$\text{span}(\underline{v_1}, \underline{v_2}) = \{ c_1 \underline{v_1} + c_2 \underline{v_2} \mid c_1, c_2 \}$



span($\underline{v_1}, \underline{v_2}$) = plane in $\mathbb{R}^3$

except: if $\underline{v_1}$ and $\underline{v_2}$ lie along the same line, i.e. $\underline{v_2} = c \cdot \underline{v_1}$ or $\underline{v_1} = c \cdot \underline{v_2}$

Ex: $\underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} -1 \\ -1/2 \end{pmatrix}$



span($\underline{v_1}, \underline{v_2}$)
= span($\underline{v_1}$)
= line in $\mathbb{R}^2$

$$\underline{v_2} = -\tfrac{1}{2}\underline{v_1} \Rightarrow c_1\underline{v_1} + c_2\underline{v_2} = c_1\underline{v_1} + c_2\left(-\tfrac{1}{2}\underline{v_1}\right)$$
$$= \left(c_1 - \tfrac{1}{2}c_2\right)\underline{v_1} = c_1' \cdot \underline{v_1}$$

2 Linear systems

Ex: Is $\underline{w} = \begin{pmatrix}1\\2\\4\end{pmatrix}$ in $\mathrm{span}(\underline{v_1}, \underline{v_2})$ with $\underline{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\underline{v_2} = \begin{pmatrix}1\\3\\9\end{pmatrix}$?

$$x_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + x_2 \cdot \begin{pmatrix}1\\3\\9\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 3x_2 &= 2 \\ x_1 + 9x_2 &= 4 \end{aligned}$$

3x2 lin. system

m×n lin. system: m eqn's (linear), n var's

Solution method: Gaussian elimination

Ex: $$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 9 & 4 \end{array}\right) \xrightarrow{\cdot(-1)} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 8 & 3 \end{array}\right) \xrightarrow{\cdot(-4)} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{array}\right)$$

coeff. matrix

augmented matrix

elementary row operations

> i) switch two rows
> ii) multiply a row with $c \neq 0$
> iii) add a multiple of a row to another row

pivot = first nonzero elem. in a row

echelon form:

- all zero rows in the bottom
- each pivot further to the right than pivots in the rows above

Lin. sys. of the echelon form:

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_2 &= 1 \\ 0 &= -1 \end{aligned}$$

no solutions

$\Downarrow$

no: $\begin{pmatrix}1\\2\\4\end{pmatrix}$ is not in $\mathrm{span}(\underline{v_1}, \underline{v_2})$

__In general:__ Any $m \times n$ lin. system can be transformed to an echelon form. The __pivot positions__ = positions of pivots in an echelon form are unique, and determine the no. of solutions:

i) pivot in last column $\rightarrow$ __no solutions__

otherwise:

ii) pivot in all variable col's $\rightarrow$ __one unique sol'n__

iii) at least one var. col. without pivot $\rightarrow$ __inf. many solutions__

$$\left.\begin{array}{l} \text{var. with pivot: basic} \\ \text{" without " : free} \end{array}\right\} \begin{array}{l} \text{no. of degrees of} \\ \text{freedom = no. free var's} \end{array}$$

__Ex:__ $x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 9 & 5 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 8 & 4 \end{array}\right) \begin{matrix} \\ \\ -4 \end{matrix} \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

__ech. form one sol'n__

Back substitution:

$$\begin{aligned} \underline{x_1} + x_2 &= 1 \\ 2\underline{x_2} &= 1 \end{aligned} \qquad \begin{aligned} x_1 &= 1/2 \\ x_2 &= 1/2 \end{aligned}$$

$$\underline{\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}}$$

__Ex:__ $x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 4 & 1 & 6 \\ 1 & -5 & 4 & 0 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & -1 & 2 \\ 0 & -6 & 2 & -4 \end{array}\right) \begin{matrix} \\ \\ 2 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

x_3 free

one deg. of freed.

$\Rightarrow$ __inf. many sol's__

$$\begin{aligned} \underline{x_1} + x_2 + 2x_3 &= 4 \\ 3\underline{x_2} - x_3 &= 2 \end{aligned}$$

$$3x_2 = 2 + x_3$$
$$\Rightarrow x_2 = 2/3 + x_3/3$$
$$x_1 = 4 - x_2 - 2x_3$$
$$= 4 - (2/3 + x_3/3) - 2x_3$$

$$\rightarrow \underline{x_1 = \frac{10}{3} - \frac{7}{3}x_3}$$

$$\underline{(x_1, x_2, x_3) = \left(\frac{10}{3} - \frac{7}{3}x_3, \frac{2}{3} + \frac{1}{3}x_3, x_3\right)}$$

with x_3 free

$x_3 = 1$: $(1,1,1)$ is one sol.
(ex.)

$$\begin{pmatrix}4\\6\\0\end{pmatrix} = 1\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\cdot\begin{pmatrix}1\\4\\-5\end{pmatrix} + 1\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}$$

Rank: A $m \times n$ matrix

$\mathrm{rk}(A)$ = no. of pivots in an echelon form of A

$$\left.\begin{array}{l} \mathrm{rk}\begin{pmatrix}1&1&2\\1&4&1\\1&-5&4\end{pmatrix} = 2 \\ \\ \mathrm{rk}\left(\begin{array}{ccc|c}1&1&2&4\\1&4&1&6\\1&-5&4&0\end{array}\right) = 2 \end{array}\right\}$$

$3-2=1$
no. of degrees of freedom

$(A|\underline{b})$

augm. matrix of $m \times n$ lin. sys.

$\mathrm{rk}(A) < \mathrm{rk}(A|\underline{b})$: no sol's

$\mathrm{rk}(A) = \mathrm{rk}(A|\underline{b})$: $n - \mathrm{rk}(A)$ = no. of degrees of freedom

③ Vector spaces

A subset $V \subseteq \mathbb{R}^n$ is called a linear subspace if

i) $\underline{v}, \underline{w}$ in $V \Rightarrow \underline{v} + \underline{w}$ in V

ii) $\underline{v}$ in V, r no. $\Rightarrow r\underline{v}$ in V

Fact: V lin. subspace of $\mathbb{R}^n$ $\Longleftrightarrow$ $V = \mathrm{span}(\underline{v_1}, \dots, \underline{v_r})$ for some n-vector $\underline{v_1} .. \underline{v_r}$.

Defn: A minimal spanning set of V is called a basis of V, and the dimension $\dim V$ of V is the no. of vectors in such a set.

If $B = \{\underline{v_1}, \dots, \underline{v_r}\}$ is a spanning set of V, i.e. $V = \mathrm{span}(\underline{v_1} \dots \underline{v_r})$, then the following are equivalent:

i) B is a <u>base</u> of V

ii) Any vector $\underline{v}$ in V can be written uniquely as

$$\underline{v} = c_1 \cdot \underline{v_1} + c_2 \underline{v_2} + \dots + c_r \underline{v_r}$$

iii) No vector in B is a linear comb. of the others

<u>Defn</u> The set B of vectors are <u>linearly independent</u> if (iii) holds, and <u>linearly dependent</u> otherwise.

<u>Ex:</u> Are $\underline{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v_2} = \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix}$ $\underline{v_3} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ lin. independent?

Look at $\underline{x_1} \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{0}$ → one solution $(0,0,0)$ ⇒ lin. independent

→ inf. many solutions ⇒ lin. dependent

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 4 & 1 \\ 1 & -5 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & -6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 free ⇒ inf. many sol's ⇒ <u>lin dependent</u>

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 3x_2 - x_3 &= 0 \end{aligned} \qquad \left. \begin{aligned} x_1 &= -x_3/3 - 2x_3 = -\tfrac{7}{3} x_3 \\ x_2 &= x_3/3 \\ x_3 &\ \text{free} \end{aligned} \right\} \qquad \begin{aligned} & x_3 = 3:\ (-7, 1, 3) \\ & -7\underline{v_1} + 1 \cdot \underline{v_2} + 3\underline{v_3} = \underline{0} \\ & \underline{v_2} = 7\underline{v_1} - 3\underline{v_3} \\ & \underline{v_3} = \tfrac{7}{3}\underline{v_1} - \tfrac{1}{3}\underline{v_2} \end{aligned}$$

$$V = \mathrm{span}(\underline{v_1}, \underline{v_2}, \underline{v_3}) = \mathrm{span}(\underline{v_1}, \underline{v_3})$$

<u>In general:</u>

$A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_r})$: $\mathrm{Col}(A) = \mathrm{span}(\underline{v_1}, \dots, \underline{v_r})$ is a lin. subspace of $\mathbb{R}^n$ of <u>dimension</u>

n-vectors $\dim \mathrm{Col}(A) = \mathrm{rk} A$, and the set of vectors corresponding to pivots

is a <u>base</u>

$A = (\underline{v_1}|\underline{v_2}|\ldots|\underline{v_r})$:
n-vectors

Null(A) = set of all solution of $A\underline{x} = \underline{0}$, the lin. sys. with augmented matrix $(A|\underline{0})$.

$= \{\underline{x} \mid A\underline{x} = \underline{0}\}$

Ex: $A = \begin{pmatrix}1&1&2\\1&4&1\\1&-5&4\end{pmatrix} \to \cdots \to \begin{pmatrix}1&1&2\\0&3&-1\\0&0&0\end{pmatrix}$

$\underline{x_1} + x_2 + 2x_3 = 0$

$3\underline{x_2} - x_3 = 0$

$(x_1, x_2, x_3) = (-\frac{7}{3}x_3, \frac{1}{3}x_3, x_3)$ $\quad x_3$ free

$= x_3 \cdot \begin{pmatrix}-7/3\\1/3\\1\end{pmatrix} = x_3 \cdot \underline{w_1}$ $\quad$ Null(A) = span$(\underline{w_1})$

In general: pivots in all col's $\Rightarrow$ one sol $\underline{x} = \underline{0}$

otherwise $\Rightarrow$ inf. many sol's

$s_1 \cdot \underline{w_1} + s_2 \cdot \underline{w_2} + \cdots + s_d \cdot \underline{w_d}$

($s_1, \ldots, s_d$ free var's)

$\Downarrow$

Null(A) is a lin. subspace of $\mathbb{R}^r$ with dim Null(A) = r - rk(A)

and $B = \{\underline{w_1}, \underline{w_2}, \ldots, \underline{w_d}\}$ is a base.

General vector space:
- lin subspace of $\mathbb{R}^n$
- abstract vector space

Remark: dim $\mathbb{R}^n = n$ and if $V \subseteq \mathbb{R}^n$ is lin subspace, then $\underline{\dim V \leq n}$.