Solutions: Self-test FORK1003 08/2019

1. $A = \begin{pmatrix} 1 & 4 & -1 & 5 & 7 \\ 3 & -1 & 2 & 2 & 4 \\ 5 & 7 & 0 & 12 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 4 & 7 \\ 3 & -1 & 4 \\ 5 & 7 & 8 \end{pmatrix}$

a)

$$\left(\begin{array}{ccccc|c} 1 & 4 & -1 & 5 & 7 & 0 \\ 3 & -1 & 2 & 2 & 4 & 0 \\ 5 & 7 & 0 & 12 & 8 & 0 \end{array}\right) \begin{array}{c} -3 \\ -5 \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & -1 & 5 & 7 & 0 \\ 0 & -13 & 5 & -13 & -17 & 0 \\ 0 & -13 & 5 & -13 & -27 & 0 \end{array}\right) -1$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & 4 & -1 & 5 & 7 & 0 \\ 0 & -13 & 5 & -13 & -17 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 \end{array}\right)$$

echelon form

(x_3, x_4 free)

$$x_1 + 4x_2 - x_3 + 5x_4 + 7x_5 = 0$$
$$-13x_2 + 5x_3 - 13x_4 - 17x_5 = 0$$
$$-10x_5 = 0$$

$-10x_5 = 0 \Rightarrow \underline{x_5 = 0}$

$$-13x_2 + 5x_3 - 13x_4 - 17x_5 = 0$$
$$\Rightarrow -13x_2 = -5x_3 + 13x_4$$
$$\underline{x_2 = \frac{5}{13}x_3 - x_4}$$

$$x_1 + 4x_2 - x_3 + 5x_4 + 7x_5 = 0$$
$$x_1 = -4x_2 + x_3 - 5x_4$$
$$= -4\left(\frac{5}{13}x_3 - x_4\right) + x_3 - 5x_4$$
$$\underline{= -\frac{7}{13}x_3 - x_4}$$

Solutions:

$(x_1, x_2, x_3, x_4, x_5) =$

$$\underline{\underline{\left(-\frac{7}{13}x_3 - x_4, \frac{5}{13}x_3 - x_4, x_3, x_4, 0\right)}}$$

with x_3, x_4 free

There are $\underline{\underline{2}}$ free variables

b) $$|B| = \begin{vmatrix} 1 & 4 & 7 \\ 3 & -1 & 4 \\ 5 & 7 & 8 \end{vmatrix} = 1\cdot(-8-28) - 4(24-20) + 7(21+5)$$
$$= -36 - 16 + 182 = \underline{\underline{130}}$$

Since B consists of cols 1, 2, 5 from A, we can also find the determinant as $|B| = 1\cdot(-13)(-10) = \underline{\underline{130}}$ from the Gaussian process in a).

$$|B| \neq 0 \Rightarrow B^{-1} = \frac{1}{|B|}\cdot \operatorname{adj}(B) = \frac{1}{130}\begin{pmatrix} -36 & -4 & 26 \\ 17 & -27 & 13 \\ 23 & 17 & -13 \end{pmatrix}^T$$

$$= \underline{\underline{\frac{1}{130}\begin{pmatrix} -36 & 17 & 23 \\ -4 & -27 & 17 \\ 26 & 13 & -13 \end{pmatrix}}}$$

c) A^2 : not defined

AB : not defined

$$BA = \underline{\underline{\begin{pmatrix} 48 & 49 & 7 & 97 & 79 \\ 20 & 41 & -5 & 61 & 49 \\ 66 & 69 & 9 & 135 & 127 \end{pmatrix}}}, \quad B^2 = \underline{\underline{\begin{pmatrix} 48 & 49 & 79 \\ 20 & 41 & 49 \\ 66 & 69 & 127 \end{pmatrix}}}$$

Since B consists of col. 1, 2, 5 from A, B^2 consists of col. 1, 2, 5 from BA.

$\underline{2.}$ $$f(x) = \frac{2}{3}\ln(1+x) + \frac{1}{3}\ln(1-x)$$

a) $$f' = \frac{2}{3}\cdot\frac{1}{1+x}\cdot 1 + \frac{1}{3}\cdot\frac{1}{1-x}\cdot(-1) = \frac{1}{3}\left(\frac{2}{1+x} - \frac{1}{1-x}\right)$$

$$= \frac{1}{3}\,\frac{2(1-x) - 1(1+x)}{(1+x)(1-x)} = \underline{\underline{\frac{1}{3}\cdot\frac{1-3x}{(1+x)(1-x)}}}$$

Stationary pts: $f'(x) = 0 \quad \frac{1}{3}\,\frac{1-3x}{1-x^2} = 0 \Rightarrow 1-3x = 0 \Rightarrow x = \underline{\underline{1/3}}$

b) $f''(x) = \left(\frac{2}{3}\frac{1}{1+x} - \frac{1}{3}\frac{1}{1-x} \right)' = \left(\frac{2}{3}(1+x)^{-1} - \frac{1}{3}(1-x)^{-1} \right)'$

$= \frac{2}{3}\cdot(-1)(1+x)^{-2}\cdot 1 - \frac{1}{3}(-1)(1-x)^{-2}\cdot(-1)$

$$= -\frac{2}{3}\frac{1}{(1+x)^2} - \frac{1}{3}\frac{1}{(1-x)^2}$$

Since $f''(x) < 0$ for all $x \in D_f = (-1,1)$, f is concave. (that is, f is concave for all $x \in D_f$).

c) f concave $\Rightarrow$ stationary pt $x = 1/3$ is global max for f

$f_{max} = f(1/3) = \frac{2}{3}\ln\left(\frac{4}{3}\right) + \frac{1}{3}\ln\left(\frac{2}{3}\right)$ is the maximum value of f

$= \frac{2}{3}(\ln 2^2 - \ln 3) + \frac{1}{3}(\ln 2 - \ln 3))$

$= \frac{2}{3}(2\ln 2 - \ln 3) + \frac{1}{3}(\ln 2 - \ln 3)$

$= \frac{5}{3}\ln 2 - \ln 3 \simeq 0{,}0566$

3.

a) $\int \frac{2-x}{x^2}\,dx = \int \frac{2}{x^2} - \frac{1}{x}\,dx = \int 2x^{-2} - \frac{1}{x}\,dx$

$= 2\cdot\frac{x^{-1}}{-1} - \ln|x| + C = -\frac{2}{x} - \ln|x| + C$

b) $\int x\cdot\ln x\,dx = \quad x^2/2\cdot\ln x - \int x^2/2\cdot 1/x\,dx$

$$\boxed{\begin{array}{ll} u = x^2/2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

int. by parts

$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

c) $\int x\sqrt{x^2+3}\,dx = \int \cancel{x}\cdot\sqrt{u}\,\frac{du}{2\cancel{x}} = \int \frac{1}{2}\sqrt{u}\,du$

$$\boxed{\begin{aligned} u &= x^2+3 \\ du &= 2x\,dx \end{aligned}}$$

substitution

$$= \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \frac{1}{2}\cdot\frac{2}{3}\,u\cdot\sqrt{u} + C$$

$$= \underline{\underline{\frac{1}{3}(x^2+3)\sqrt{x^2+3} + C}} = \underline{\underline{\frac{1}{3}(x^2+3)^{3/2} + C}}$$

d) $\int \frac{6}{4-x^2}\,dx = \int \frac{6}{(2-x)(2+x)}\,dx = \int \frac{3/2}{2-x} + \frac{3/2}{2+x}\,dx$

Partial fractions:

$$\frac{6}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x} \quad \Big|\cdot(2-x)(2+x)$$

$$6 = A(2+x) + B(2-x)$$
$$= (2A+2B) + (A-B)x$$

$2A+2B=6$ $\quad 4A=6 \Rightarrow A = 6/4 = 3/2$

$A-B=0$ $\quad B=A$ $\quad \underline{B=3/2}$

$$= \frac{3}{2}\ln|2-x|\cdot\frac{1}{(-1)} + \frac{3}{2}\ln|2+x|\cdot\frac{1}{1} + C$$

$$= \underline{\underline{\frac{3}{2}\ln|2+x| - \frac{3}{2}\ln|2-x| + C}}$$

$$= \underline{\underline{\frac{3}{2}\ln\left|\frac{2+x}{2-x}\right| + C}}$$

<u>4.</u> $f(x,y) = x^3 - 3xy + y^3$

a) $f'_x = \underline{\underline{3x^2-3y}}$

$f'_y = \underline{\underline{-3x+3y^2}}$

<u>Stationary pts:</u> $3x^2-3y=0$, $-3x+3y^2=0$

(1) $3y = 3x^2 \Rightarrow \underline{y=x^2}$

(2) $-3x+3(x^2)^2=0$

$-3x+3x^4=0$

$-3x(1-x^3)=0$

$\underline{x=0}$ or $x=\sqrt[3]{1}=\underline{1}$

$\underline{y=0}$ $\quad \underline{y=1}$

<u>Stat. pts:</u> $(x,y) = \underline{\underline{(0,0),(1,1)}}$

b) $H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ for general pt. (x,y)

$(0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

$\det = 0 - 9 = -9 < 0$

$(0,0)$ saddle pt

$(1,1)$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$

$\det = 36 - 9 = 27 > 0$

$A = 6 > 0$

$(1,1)$ local min.

c) Difficult ! max $f(x,y) = x+y$ when $x^3 - 3xy + y^3 = 0$

i) Find candidate pts : (6p)

$L = x + y - \lambda(x^3 - 3xy + y^3)$

$$\boxed{\begin{aligned} L'_x &= 1 - \lambda \cdot (3x^2 - 3y) = 0 \\ L'_y &= 1 - \lambda\,(-3x + 3y^2) = 0 \\ & x^3 - 3xy + y^3 = 0 \end{aligned}} \quad \begin{matrix} \left.\right\} \text{FOC} \\ \left.\right\} \text{C} \end{matrix}$$

(1) $1 = \lambda \cdot (3x^2 - 3y) = \lambda \cdot 3(x^2 - y)$

$\Rightarrow \lambda = \dfrac{1}{3(x^2 - y)}$

note: we divide by $x^2 - y$, but there are no sol's with $x^2 = y$; this would give:

$1 - \lambda(3x^2 - 3y) = 0$

$1 - \lambda \cdot 0 = 0$

$1 = 0$

which is impossible

(2) $1 - \dfrac{1}{3(x^2 - y)} \cdot (-3x + 3y^2) = 0$

$1 - \dfrac{\cancel{3}(y^2 - x)}{\cancel{3}(x^2 - y)} = 0$

$\dfrac{y^2 - x}{x^2 - y} = 1 \Rightarrow y^2 - x = x^2 - y$

$y^2 - x^2 = x - y$

$$x-y = y^2-x^2 = (y-x)(y+x) = -(x-y)(x+y)$$

$$x-y + (x-y)(x+y) = 0$$

$$(x-y)(1+x+y) = 0$$

a) $\underline{x=y}$ or b) $\underline{x+y=-1}$

Case a)

(3) $x^3-3xy+y^3=0$

$y^3-3y^2+y^3=0$

$2y^3-3y^2=0$

$y^2(2y-3)=0$

$\underline{y=0}$ or $\underline{y=3/2}$

$x=0$ $\quad$ $x=3/2$

$\Downarrow$

$(0,0)$, $(3/2, 3/2)$

<u>Find λ from (1):</u>

$(x,y)=(0,0) \Rightarrow$

(1) gives $1-\lambda\cdot 0=0$
no solution

$(x,y)=(3/2,3/2) \Rightarrow$

$$\lambda = \frac{1}{3(9/4-3/2)} = \frac{1}{3\cdot 3/4} = \underline{\frac{4}{9}}$$

$\Downarrow$

<u>One solution in a)</u>

$(x,y;\lambda) = \underline{(3/2, 3/2; 4/9)}$

with $f=x+y=\underline{3}$

Case b)

$y=-1-x$

(3) $x^3-3xy+y^3=0$

$x^3-3x(-1-x)+(-1-x)^3=0$

$\cancel{x^3}+\cancel{3x}+\cancel{3x^2}-1-\cancel{3x}-\cancel{3x^2}-\cancel{x^3}=0$

$-1=0$

not possible,
no solutions in b)

$\downarrow$

<u>One candidate pt:</u>

$(x;y;\lambda) = \underline{\underline{(3/2, 3/2; 4/9)}}$

with $\underline{\underline{f=3}}$

ii) <u>Check if the cond. pt. is max:</u> 6p.

$f(3/2, 3/2) = \underline{3}$ Are there any pts on $x^3-3xy+y^3=0$ with $f>3$?

Check if $f = \underline{x+y=a}$ intersects $\underbrace{x^3-3xy+y^3=0}_{\text{constraint}}$ with $a>3$:

$\underline{x+y=a} \Rightarrow y = a-x$

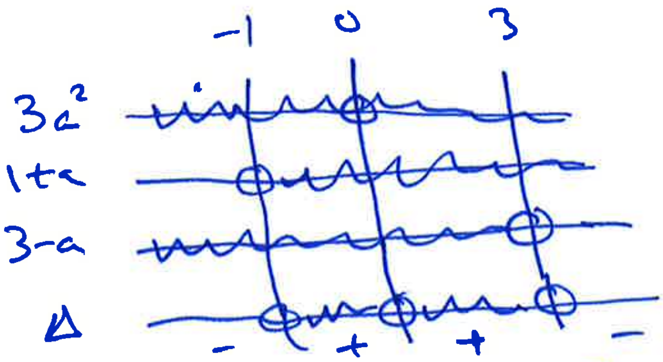
$$x^3-3xy+y^3=0$$
$$x^3-3x(a-x)+(a-x)^3=0$$
$$\cancel{x^3}-\underline{3xa}+\underline{3x^2}+a^3-\underline{3a^2x}+\underline{3ax^2}-\cancel{x^3}=0$$
$$(3+3a)x^2+(-3a-3a^2)x+a^3=0$$

quadratic eqn.

<u>note:</u> It has sol's if and only if $\Delta = B^2-4AC$ by the ABC-formula

($\Delta = B^2-4AC$ is the expression under the square root)

$$\underline{\Delta} = B^2-4AC = (-3a-3a^2)^2-4(3+3a)a^3$$
$$= (-3a)^2(1+a)^2-4a^3\cdot 3\cdot(1+a)$$
$$= 3a^2(1+a)\left[3(1+a)-4a\right]$$
$$= 3a^2(1+a)(3-a)$$



$\Delta \geq 0$ for $-1 \leq a \leq 3$

$\Delta < 0$ for $a>3$ $\Rightarrow$

no soln. (intersection pt.) if $a>3$.

<u>Conclusion:</u>

$f_{max} = \underline{\underline{3}}$ at $(3/2, 3/2)$ is the <u>max</u>