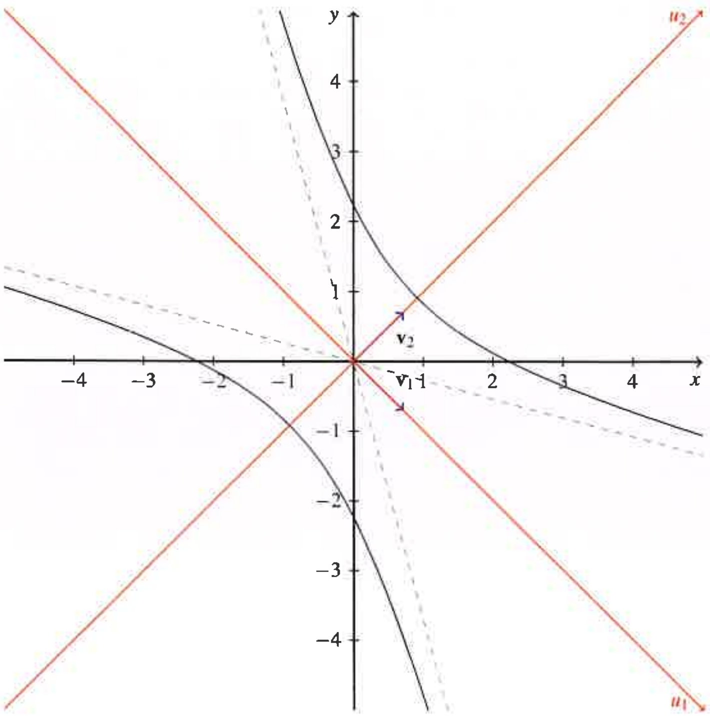
$$(\sqrt{3}u_2 - u_1)(\sqrt{3}u_2 + u_1) = 5$$

If we define new coordinates (v_1, v_2) by $v_1 = \sqrt{3}u_2 - u_1$ and $v_2 = \sqrt{3}u_2 + u_1$, this equation is the hyperbola $v_1 \cdot v_2 = 5$, or $v_2 = 5/v_1$. The curve is shown in the figure below with the (u_1, u_2) coordinate system marked. The vectors $\mathbf{v}_1, \mathbf{v}_2$ are the column vectors of P, and the dashed lines are the asymptotes $\sqrt{3}u_2 \pm u_1 = 0$.



1.5 Unconstrained optimization

Solution 1.5.1. In each case, we compute the first order partial derivatives and solve the first order conditions to find the stationary points. Then we compute the Hessian and determine the definiteness of the Hessian at each stationary point to classify it.

a) The first order conditions (FOC) are given by

$$f'_x = 4x^3 + 2x - 6y = 0, \quad f'_y = -6x + 6y = 0$$

From the second equation $x = y$, and the first then gives $4x^3 - 4x = 4x(x^2 - 1) = 0$, or $x = 0$ or $x = \pm 1$. The stationary points are $(0,0)$, $(1,1)$ and $(-1,-1)$. The Hessian matrix is given by

$$H(f) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}$$

At $(0,0)$ we have $D_1 = 2$ and $D_2 = 12 - 36 = -24$. This point is a saddle point since D_2 is negative. At $(\pm 1, \pm 1)$ we have $D_1 = 14$ and $D_2 = 84 - 36 = 48$. These points are local minima since $D_1, D_2 > 0$.

b) The first order conditions (FOC) are given by

$$f'_x = 2x - 6y + 10 = 0, \quad f'_y = -6x + 4y + 2 = 0$$

We combine the equations to get $-14y + 32 = 0$ or $y = 32/14 = 16/7$, and then $x = 3(16/7) - 5 = 13/7$. The unique stationary point is $(13/7, 16/7)$. The Hessian matrix is given by

$$H(f) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}$$

and $D_1 = 2$, $D_2 = 8 - 36 = -28$. The stationary point is a saddle point since D_2 is negative.

c) The first order conditions (FOC) are given by

$$f'_x = y^2 + 3x^2y - y = y(y + 3x^2 - 1) = 0, \quad f'_y = 2xy + x^3 - x = x(2y + x^2 - 1) = 0$$

From the second equation either $x = 0$ or $x^2 = 1 - 2y$. In the first equation, $x = 0$ gives $y^2 - y = 0$, that is $y = 0$ or $y = 1$, and the stationary points in this case are $(0,0)$ and $(0,1)$. The other possibility is $x \neq 0$ but $x^2 = 1 - 2y$. In this case, the first equation gives $y = 0$ or $y + 3x^2 - 1 = 0$. If $y = 0$ then $x = \pm 1$, and if $y = 1 - 3x^2$, then $x^2 = 1 - 2(1 - 3x^2) = 6x^2 - 1$, or $x = \pm 1/\sqrt{5}$. This gives stationary points $(\pm 1, 0)$ and $(\pm 1/\sqrt{5}, 2/5)$. The Hessian matrix is given by

$$H(f) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}$$

At $(0,0)$ and $(0,1)$, we have $D_1 = 0$ and $D_2 = -1$. These points are saddle points since D_2 is negative. At $(\pm 1, 0)$ we have $D_1 = 0$ and $D_2 = -4$. These points are also saddle points since D_2 is negative. At $(\pm 1/\sqrt{5}, 2/5)$, we have $D_1 = 12x/5$ and $D_2 = 12(1/5)(2/5) - (4/5 + 3/5 - 1)^2 = 20/25$. This is positive definite when x is positive, and negative definite when x is negative. Therefore $(1/\sqrt{5}, 2/5)$ is a local minimum and $(-1/\sqrt{5}, 2/5)$ is a local maximum

d) The first order conditions (FOC) are given by

$$f'_x = 12x^3 + 6xy = 0, \quad f'_y = 3x^2 - 3y^2 = 0$$

From the second equation $x^2 = y^2$, and therefore $x = y$ or $x = -y$. If $x = y$, the first equation gives $12x^3 + 6x^2 = 6x^2(2x+1) = 0$, that is $x = 0$ or $x = -1/2$. This gives stationary points $(0,0)$ and $(-1/2,-1/2)$. If $x = -y$ then the first equation gives $12x^3 - 6x^2 = 6x^2(2x-1)$, that is $x = 0$ or $x = 1/2$. This gives stationary points $(0,0$ (again) and $(1/2,-1/2)$. The Hessian matrix is given by

$$H(f) = \begin{pmatrix} 36x^2+6y & 6x \\ 6x & -6y \end{pmatrix}$$

At $(\pm 1/2,-1/2)$ we have $D_1 = 6$ and $D_2 = 18-9 = 9$. These points are local minima since D_1, D_2 are positive. At $(0,0)$ the Hessian matrix is the zero matrix, and the second derivative test is inconclusive. Along the path where $x = 0$ and $y = a$ (the y-axis), we have that $f(0,a) = -a^3$. Since $f(0,a)$ is negative for $a > 0$ and positiv for $a < 0$, the stationary point $(0,0)$ at $a = 0$ is a saddle point.

e) See Solution 5.2 a).

f) See Solution 5.2 b).

g) The first order conditions (FOC) are given by

$$f'_x = 2x+6y-10 = 0, \quad f'_y = 6x+2y-3z-5 = 0, \quad f'_z = -3y+8z-21 = 0$$

The unique stationary point is $(x,y,z) = (2,1,3)$, and we find it by solving the linear system (for instance using Gaussian elimination). The Hessian is indefinite at this stationary point since the Hessian matrix

$$H(f) = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$$

has leading principal minors $D_1 = 2$ and $D_2 = -32$ (there is no need to compute the last leading principal minor). The stationary point $(2,1,3)$ is a saddle point for f.

Solution 1.5.2. In each case, we compute the first order partial derivatives and solve the first order conditions to find the stationary points. Then we compute the Hessian and determine the definiteness of the Hessian at each stationary point to classify it.

a) The function $f(x,y) = e^{xy}$ has partial derivatives $f'_1 = ye^{xy}$ and $f'_2 = xe^{xy}$. Hence the stationary points are given by $ye^{xy} = xe^{xy} = 0$, and we see that there is a unique stationary point $(x,y) = (0,0)$. The Hessian matrix is

$$H(f)(0,0) = \begin{pmatrix} y^2e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2e^{xy} \end{pmatrix}(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This matrix is indefinite since $D_2 = -1$, and $(0,0)$ is a saddle point.

b) The function $f(x,y) = \ln(x^2+y^2+1)$ has partial derivatives $f'_1 = 2x/(x^2+y^2+1)$ and $f'_2 = 2y/(x^2+y^2+1)$. Hence there is a unique stationary point $(x,y) = (0,0)$. The Hessian matrix is

$$H(f)(0,0) = \begin{pmatrix} \frac{2x^2+2y^2+2}{(x^2+y^2+1)^2} & \frac{4xy}{(x^2+y^2+1)^2} \\ \frac{-4xy}{(x^2+y^2+1)^2} & \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Since $D_1 = 2$, $D_2 = 4$, the Hessian matrix is posive definte, and $(0,0)$ is a local minimum point for f.

Solution 1.5.3. In each case, we find the Hessian matrix $H(f)$ at a general point, and use its definiteness to determine whether f is convex or concave:

a) When $f(x,y) = x^4 + x^2 - 6xy + 3y^2$, the Hessian matrix at a general point is

$$H(f) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}$$

Since $D_1 = 12x^2 + 2 > 0$ and $D_2 = (12x^2+2)\cdot 6 - 36 = 72x^2 - 24$ at a general point, it follows that $H(f)$ can indefinite, for instance at $(0,0)$ with $D_2(0,0) = -24 < 0$. Hence f is neither convex nor concave.

b) When $f(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y$, the Hessian matrix at a general point is

$$H(f) = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}$$

Since $D_2 = 8 - 36 = -28 < 0$ at a general point, it follows that f is neither convex nor concave.

c) When $f(x,y) = xy^2 + x^3y - xy$, the Hessian matrix at a general point is

$$H(f) = \begin{pmatrix} 6xy & 2y+3x^2-1 \\ 2y+3x^2-1 & 2x \end{pmatrix}$$

Since $D_1 = 6xy$ can take both positive and negative values, it follows that f is neither convex nor concave.

d) When $f(x,y) = 3x^4 + 3x^2y - y^3$, the Hessian matrix at a general point is

$$H(f) = \begin{pmatrix} 36x^2+6y & 6x \\ 6x & -6y \end{pmatrix}$$

Since $D_1 = 36x^2 + 6y$ can take both positive and negative values, for instance $D_1(1,0) = 36 > 0$ and $D_1(0,-1) = -6 < 0$, it follows that f is neither convex nor concave.

e) When $f(x,y) = e^{xy}$, the Hessian matrix at a general point is

$$H(f) = \begin{pmatrix} y^2e^{xy} & (1+xy)e^{xy} \\ (1+xy)e^{xy} & x^2e^{xy} \end{pmatrix}$$

Since $H(f)(x,y)$ is indefinite at the point $(x,y) = (0,0)$, with $D_2 = 0^2 - 1^2 = -1$, it follows that f is neither convex nor concave.

f) When $f(x,y) = \ln(x^2 + y^2 + 1)$, the Hessian matrix at a general point is

Solution 1.6.3. In each case, we use the first order conditions (FOC's) to find candidate points, and check whether the candidate points are optimal:

a) The Lagrangian of the optimization problem max $f(x,y) = xy$ when $2x+3y = 12$ is given by $L = xy - \lambda(2x+3y-12)$, and we have FOC's

$$L'_x = y - 2\lambda = 0 \quad L'_y = x - 3\lambda = 0$$

When we solve the FOC's, we get $y = 2\lambda$ and $x = 3\lambda$, and when we substitute this into the constraint $2x+3y = 12$, we get $12\lambda = 12$, or $\lambda = 1$. This gives the candidate $(x,y;\lambda) = (3,2;1)$ for max with $f(3,2) = 6$. Moreover, NDCQ is satisfied at all points since

$$\text{rk}\begin{pmatrix}2 & 3\end{pmatrix} = 1$$

To see that $(3,2)$ is a maximum point with $f(3,2) = 6$, we cannot use the SOC (second order condition) since the Lagrangian is not concave, or the EVT (Extreme Value Theorem) since the domain is not bounded. Instead, we solve the constraint for x, to get $x = (12-3y)/2$, and substitute this into

$$f(x,y) = xy = \frac{12-3y}{2}y = 6y - 3y^2/2$$

We can consider $f(y) = 6y - 3y^2/2$ as a function in one variable, with y arbitrary; this is an unconstrained problem. The stationary point is given by $f'(y) = 6 - 3y = 0$, and is at $y = 2$, and it is a maximum point since $f''(y) = -3/2$ so that f is concave in y. Hence $(x,y) = (3,2)$ is a maximum point with $f_{\max} = f(3,2) = 6$.

b) The optimization problem max $f(x,y) = x^2y$ when $2x^2+5y^2 = 15$ has Lagrangian $L = x^2y - \lambda(2x^2+5y^2-15)$, and we have FOC's

$$L'_x = 2xy - \lambda\cdot 4x = 0 \quad L'_y = x^2 - \lambda\cdot 10y = 0$$

We solve the FOC's, and get $x(2y-4\lambda) = 0$ from the first equation, and this gives $x = 0$ or $y = 2\lambda$. When $x = 0$, the constraint $2x^2+5y^2 = 15$ gives $y = \pm\sqrt{3}$, and then $\lambda = 0$ by the second equation. When $y = 2\lambda$, then the second equation gives $x^2 = 20\lambda^2$, or $x = \pm\sqrt{20}\lambda$, and the constraint gives $60\lambda^2 = 15$, or $\lambda = \pm 1/2$. We have the candidates

$$(x,y;\lambda) = (0,\pm\sqrt{3};0),\ (\pm 5,1,1/2),\ (\pm 5,-1,-1/2)$$

The points $(x,y) = (\pm 5,1)$ with $\lambda = 1/2$ are the best candidates for maximum points, with $f(\pm 5,1) = 5$. Moreover, NDCQ is satisfied at all admissible points since

$$\text{rk}\begin{pmatrix}4x & 10y\end{pmatrix} = 1$$

for all points except $(x,y) = (0,0)$, and this point does not satisfy the constraint $2x^2+5y^2 = 15$. Since $D = \{(x,y) : 2x^2+5y^2 = 15\}$ is compact (an ellipse), there is a max by the EVT, and $(x,y) = (\pm 5,1)$ is the maximum points with $f_{\max} = f(\pm\sqrt{5},1) = 5$.