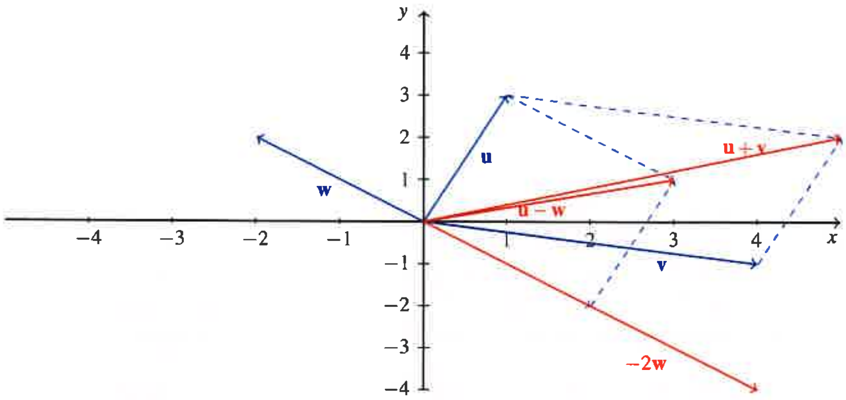
1.2 Vectors and vector spaces

Solution 1.2.1. The vectors are shown in the figure below. In coordinates, we have that $\mathbf{u}+\mathbf{v} = (5,2)$, $\mathbf{u}-\mathbf{w} = (3,1)$, and $-2\mathbf{w} = (4,-4)$.



Solution 1.2.2. When $\mathbf{u} = (1,3)$, $\mathbf{v} = (4,-1)$ and $\mathbf{w} = (-2,2)$, we get:

a) $\|\mathbf{u}\| = \sqrt{1^2+3^2} = \sqrt{10}$

b) $\|\mathbf{v}\| = \sqrt{4^2+(-1)^2} = \sqrt{17}$

c) $\|\mathbf{w}\| = \sqrt{(-2)^2+2^2} = \sqrt{8}$

d) $\mathbf{u}\cdot\mathbf{v} = 1\cdot 4+3\cdot(-1) = 1$

e) $\mathbf{u}\cdot\mathbf{w} = 1\cdot(-2)+3\cdot 2 = 4$

f) $\mathbf{v}\cdot\mathbf{w} = 4\cdot(-2)+(-1)\cdot 2 = -10$

Solution 1.2.3. We use the inner products from the previous problem and get:

a) $\text{Proj}_{\mathbf{u}}(\mathbf{v}) = \frac{1}{10}(1,3)$

b) $\text{Proj}_{\mathbf{u}}(\mathbf{w}) = \frac{4}{10}(1,3)$

c) $\text{Proj}_{\mathbf{v}}(\mathbf{w}) = \frac{-10}{17}(4,-1)$

d) $\text{Proj}_{\mathbf{v}}(\mathbf{u}) = \frac{1}{17}(4,-1)$

Solution 1.2.4. To express $(8,9)$ as a linear combination of $(2,5)$ and $(-1,3)$, we consider the vector equation

$$x\cdot(2,5)+y\cdot(-1,3) = (8,9)$$

This results in a 2×2 linear system, and we solve it by Gaussian elimination:

$$\left(\begin{array}{cc|c} 2 & -1 & 8 \\ 5 & 3 & 9 \end{array}\right) \quad\to\quad \left(\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & 11/2 & -11 \end{array}\right)$$

Back substitution gives $11y/2 = -11$, or $y = -2$, and $2x-(-2) = 8$, or $x = 3$. Hence $3\mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{w}$.

Solution 1.2.5. We find the following parametric descriptions (others are possible):

c) Since there are two free variables, $\dim \mathrm{Null}(A) = 2$. To find a base, we use the echelon form from a) and back substitution. This gives $x_2 = 12x_3 - 7x_4$ and $x_1 = -2x_2 - 4x_3 + x_4 = -28x_3 + 15x_4$. We can write the solutions in terms of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -28x_3 + 15x_4 \\ 12x_3 - 7x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -28 \\ 12 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 15 \\ -7 \\ 0 \\ 1 \end{pmatrix}$$

Hence $W = \mathrm{Span}(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1 = (-28, 12, 1, 0)$ and $\mathbf{w}_2 = (15, -7, 0, 1)$. These vectors are linearly independent since the variables x_3, x_4 are free, and this means that $B = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a base of $W = \mathrm{Null}(A)$.

Solution 1.2.16. Let $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r)$, and let A be the matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ as columns.

a) We find an echelon form of A, and identify the pivot positions. The vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ that correspond to pivot positions form a base B for V. The other vectors correspond to free variables. Each of these vectors can be expressed as linear combinations of the vectors in B.

b) The dimension of V is the number of vectors in a base, and there is one base vector for each pivot position. Since the rank of A is the number of pivot positions, we have that $\dim V = \mathrm{rk}(A)$.

1.3 Matrices and determinants

Solution 1.3.1. We compute the matrix products:

a) $$AB = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 6 & 5 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

b) $$BA = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 5 \\ 2 & 0 & 2 \\ 2 & 3 & 5 \end{pmatrix}$$

c) $$A^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 7 & 1 & 13 \\ 1 & -2 & -2 \end{pmatrix}$$

d) $$B^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

e) $$(A+B)^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 6 & 14 \\ 16 & 8 & 19 \\ 4 & 4 & 5 \end{pmatrix}$$

f) $ABC = AB \cdot C = \begin{pmatrix} 2 & 2 & 2 \\ 6 & 5 & 3 \\ 0 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 8 \\ 6 & 1 & 17 \\ 0 & -2 & 2 \end{pmatrix}$

In e) we could also have used that $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ and the products computed in a) - d).

Solution 1.3.2. The matrix A is symmetric and the matrix B is not symmetric:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 7 & 1 \\ 3 & 7 & 1 & 4 \\ 4 & 1 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 7 & 5 \\ 2 & 6 & 2 & 4 \\ 4 & 1 & 4 & 3 \end{pmatrix} \neq B^T$$

Solution 1.3.3. We have

a) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C = AB^2C - ABCB + CABC - AB^2C + CA^2C - CABC = -ABCB + CA^2C$

b) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2 = AC - BC - A^2 + BA + CA - BA - C^2 + BC + C^2 - CA - AC + A^2 = 0$

Solution 1.3.4. We find:

a) The linear system is

$$\begin{aligned} 3x + y + 5z &= 4 \\ 5x - 3y + 2z &= -2 \\ 4x - 3y - z &= -1 \end{aligned}$$

b) We have that A is invertible since $|A| \neq 0$:

$$|A| = \begin{vmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{vmatrix} = 3(3+6) - 1(-5-8) + 5(-15+12) = 27 + 13 - 15 = 25$$

The inverse matrix is therefore given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{25} \begin{pmatrix} 9 & 13 & -3 \\ -14 & -23 & 13 \\ 17 & 19 & -14 \end{pmatrix}^T = \frac{1}{25} \begin{pmatrix} 9 & -14 & 17 \\ 13 & -23 & 19 \\ -3 & 13 & -14 \end{pmatrix}$$

c) Since A is invertible, the linear system has one unique solution.

Solution 1.3.5. We have

$$A^T A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 15 & 14 \\ 0 & 5 & 0 & 3 \\ 15 & 0 & 45 & 42 \\ 14 & 3 & 42 & 41 \end{pmatrix}$$