Solutions for Problem Set 2

1.

a) $f'(x) = 4x^3 - \frac{3}{2}x^{1/2} + 3$

$= 4x^3 - \frac{3}{2}\sqrt{x} + 3$

$f''(x) = 12x^2 - \frac{3}{4}x^{-1/2}$

$= 12x^2 - \frac{3}{4\sqrt{x}}$

b) $u = x^2+1$

$f'(x) = 1\cdot\sqrt{u} + x\cdot\frac{1}{2}u^{-1/2}\cdot 2x$

$= \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}$

$f''(x) = \frac{1}{2}u^{-1/2}\cdot 2x + \left(x^2\cdot u^{-1/2}\right)'$

$\left(\sqrt{u} + \frac{x^2}{\sqrt{u}}\right)' = \frac{x}{\sqrt{u}} + 2x\cdot u^{-1/2} + x^2\cdot\left(-\frac{1}{2}\right)u^{-3/2}\cdot 2x$

$\left(u^{1/2} + x^2u^{-1/2}\right)' = \frac{3x}{\sqrt{u}} - \frac{x^3}{u\sqrt{u}} = \frac{3x(x^2+1)-x^3}{u\sqrt{u}}$

$= \frac{2x^3+3x}{(x^2+1)\sqrt{x^2+1}}$

c) $f'(x) = \left(\frac{x+1}{x^2-3x+2}\right)' = \frac{1\cdot(x^2-3x+2) - (x+1)(2x-3)}{(x^2-3x+2)^2}$

$= \frac{x^2-3x+2-(2x^2-x-3)}{v^2} = \frac{-x^2-2x+5}{(x^2-3x+2)^2}$

$f''(x) = \frac{(-2x-2)v^2 - (-x^2-2x+5)\cdot 2v\cdot(2x-3)}{v^4}$

$= \frac{(-2x-2)(x^2-3x+2) - 2(-x^2-2x+5)(2x-3)}{v^3}$

$= \frac{(-2x^3+4x^2+2x-4) - (-4x^3-2x^2+32x-30)}{v^3}$

$= \frac{2x^3+6x^2-30x+26}{(x^2-3x+2)^3}$

d) $f'(x) = 1\cdot e^x + xe^x - (2xe^{-x} + x^2e^{-x}\cdot(-1)) + e^{2x-1}\cdot 2$

$= (x+1)e^x + (x^2-2x)e^{-x} + 2e^{2x-1}$

$f''(x) = 1e^x + (x+1)e^x + (2x-2)e^{-x} + (x^2-2x)e^{-x}\cdot(-1) + 2e^{2x-1}\cdot 2$

$$= (x+2)e^x + (-x^2+4x-2)e^{-x} + 4e^{2x-1}$$

e) $f'(x) = \frac{1}{x} - \frac{1}{x-1} = \frac{(x-1)-x}{x(x-1)} = \frac{-1}{x(x-1)}$

$f''(x) = (x^{-1} - (x-1)^{-1})' = -1 \cdot x^{-2} - (-1)(x-1)^{-2} \cdot 1$

$= -\frac{1}{x^2} + \frac{1}{(x-1)^2} = \frac{-(x-1)^2 + x^2}{x^2(x-1)^2}$

$= \frac{2x-1}{x^2(x-1)^2}$

f) $f'(x) = \frac{1}{x^3-x^2} \cdot (3x^2-2x)$

$= \frac{x(3x-2)}{x^2(x-1)} = \frac{3x-2}{x(x-1)}$

$f''(x) = \frac{3x(x-1) - (3x-2)(2x-1)}{x^2(x-1)^2}$

$= \frac{3x^2-3x-(6x^2-7x+2)}{x^2(x-1)^2}$

$= \frac{-3x^2+4x-2}{x^2(x-1)^2}$

2.

a) $f'(x)=2x \quad f''(x)=2>0$ for all x

$\Rightarrow$ f convex

b) $f'(x)=ax^{a-1} \quad f''=a(a-1)x^{a-2}$

$f(x)=x^a$ $(a>0)$ defined for $x>0$

$\Rightarrow x^{a-2}>0$, $a>0$

$a>1$: $f''(x)>0 \Rightarrow$ f convex

$0<a<1$: $f''(x)<0$ f concave

c) $f'(x)=f''(x)=e^x>0$ f convex

d) $f'(x)=\frac{1}{x}=x^{-1} \quad f''(x)=-1\cdot x^{-2}=-1/x^2<0$

f concave

e) $f''(x)=\dfrac{-3x^2+4x-2}{x^2(x-1)^2}$

$x^2, (x-1)^2>0$

$-3x^2+4x-2=0 \Rightarrow x=\dfrac{-4\pm\sqrt{16-4\cdot(-3)(-2)}}{2\cdot(-3)}$

$=\dfrac{-4\pm\sqrt{-8}}{-6}$ no solution

$-3x^2+4x-2<0$ for all x

$\Rightarrow f''(x)<0$ for all x f concave