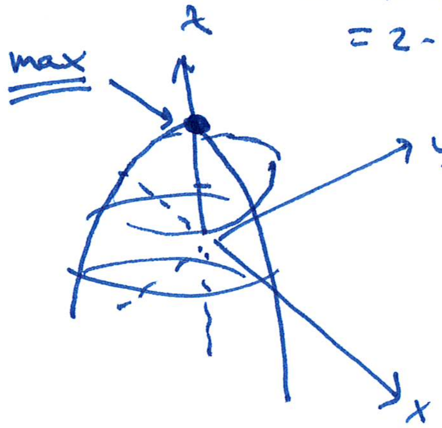
Plan

1 Functions in two variables and partial derivatives
2 Unconstrained optimization
3 Constrained optimization and Lagrange multipliers

① Functions in two variables

Ex: $f(x,y) = 2 - x^2 - y^2$

$z = f(x,y) = 2 - x^2 - y^2$



Level curves: $z = c$

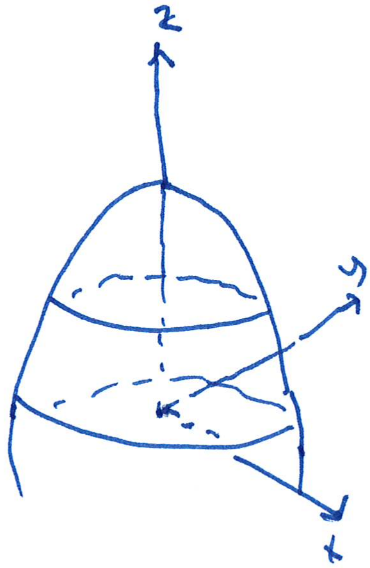
$$2 - x^2 - y^2 = c$$
$$2 - c = x^2 + y^2$$
$$\boxed{x^2 + y^2 = 2 - c}$$

$x^2 + y^2 = r^2$
circle,
center $(0,0)$
radius r

$c = 2$: $x^2 + y^2 = 0$ point $(0,0)$

$c > 2$: $x^2 + y^2 < 0$ no points

$c < 2$: $x^2 + y^2 = 2 - c > 0$ circle, $r = \sqrt{2-c}$



Partial derivatives:

Ex:

$$f'_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = (2 - x^2 - y^2)'_x = \underline{-2x}$$

$$f'_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} = (2 - x^2 - y^2)'_y = \underline{-2y}$$

Defn. A stationary pt for $f(x,y)$ is a point such that

$$\boxed{f'_x = 0, \; f'_y = 0}$$

FOC = first order condition

Ex: $f = 2 - x^2 - y^2$

$$\left.\begin{aligned} f'_x &= -2x = 0 \\ f'_y &= -2y = 0 \end{aligned}\right\} \quad \begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$$

Stationary pts:

$(x,y) = \underline{\underline{(0,0)}}$

If a point (x^*, y^*) is a max/min for $f(x,y)$, then either

critical pts
- i) (x^*, y^*) is a stationary pt of f
- ii) (x^*, y^*) is a point where f'_x or f'_y is not defined
- iii) (x^*, y^*) is a boundary pt of the adm. region

② Unconstrained optimization

$$\boxed{\max/\min \; f(x,y)}$$

Method:

(a) Find all candidate points = find all stationary pts (if f is "nice")

(b) Classify the stationary points as local max/local min/saddle pts. using the second derivative test.

(c) Somehow figure out if local max/min are also global max/min.

Ex: $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y \cdot 1 + 0 = \underline{3x^2 - 3y} = 0$

$f'_y = 0 - 3x \cdot 1 + 3y^2 = \underline{-3x + 3y^2} = 0$

$\frac{3x^2}{3} = \frac{3y}{3} \Rightarrow y = x^2$

$-3x + 3(x^2)^2 = 0$

$-3x + 3x^4 = 0 \qquad 3x(-1 + x^3) = 0$

$3x = 0$ or $x^3 = 1$

$\underline{x=0} \qquad x = \sqrt[3]{1} = 1$

$\underline{y=0} \qquad \underline{x=1}$

$\underline{y=1}$

Stationary pts: $(x,y) = \underline{(0,0)}, \underline{(1,1)}$ $\qquad f(0,0) = 0$, $f(1,1) = -1$

$f'_x = 3x^2 - 3y$
$f'_y = -3x + 3y^2$

$$\left.\begin{matrix} f''_{xx} = 6x & f''_{xy} = -3 \\ f''_{yx} = -3 & f''_{yy} = 6y \end{matrix}\right\} H(f)(x,y) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \underline{\begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}}$$

Defn: The Hessian matrix of $f(x,y)$ is

$$H(f)(x,y) = \begin{pmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{pmatrix}$$

$f''_{xy} = f''_{yx}$

It is a 2×2-matrix, and it is symmetric if f is "nice".

Second derivative test

If (x^*, y^*) is a stationary point for $f(x,y)$, then we have:

① If $|H(f)(x^*,y^*)| > 0$ and $\operatorname{tr} H(f)(x^*,y^*) > 0$, then it is local min

② If — " — $\operatorname{tr} H(f)(x^*,y^*) < 0$ — " — local max

③ If $|H(f)(x^*,y^*)| < 0$, then it is saddle point.

If $|H(f)(x^*,y^*)| = 0$, then the test is inconclusive.

Explanation:

$$H(f)(x^*,y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} \qquad \begin{aligned} |H(f)(x^*,y^*)| &= AC - B^2 \\ \operatorname{tr} H(f)(x^*,y^*) &= A + C \end{aligned}$$

Alternative description:

$A>0 \Leftrightarrow A, C > 0$

$AC - B^2 > 0$, $A + C > 0$ $\Rightarrow$ local min

$AC - B^2 > 0$, $A + C < 0$ $\Rightarrow$ local max

$AC - B^2 < 0$ $\Rightarrow$ saddle point

Note:

If $AC - B^2 > 0$, then $AC > B^2$

$\Rightarrow AC > 0$

$\Updownarrow$

$A, C > 0$

or

$A, C < 0$

Ex: $f(x,y) = x^3 - 3xy + y^3$

$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

Stat. pts: $(0,0)$, $(1,1)$

$f=0$ $f=-1$

Classification:

i) $(0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\det = 0-(-3)^2 = -9 < 0$

$\Rightarrow$ $(0,0)$ saddle pt

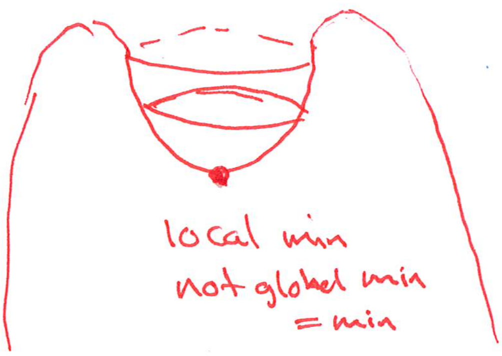
ii) $(1,1)$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\det = 36 - 9 = 27 > 0$

$\mathrm{tr} = 6+6 = 12 > 0$

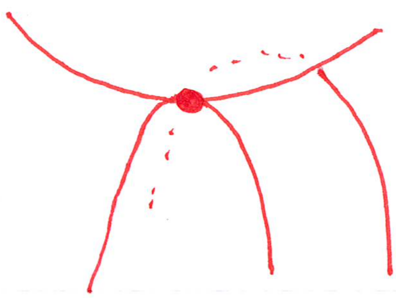
$\Rightarrow$ $(1,1)$ local min

Concl. so far: no max

$f(1,1) = -1$ local min maybe min?



Defn: A saddle point is a stationary point that is not local max or local min.



Note:

$A = f''_{xx}(x^*, y^*)$

$C = f''_{yy}(x^*, y^*)$

$f(x,y) = x^3 - 3xy + y^3$

$y=0$: $f(x,0) = x^3$

$x = -2$

$f(-2,0) = -8 < -1$

$\Big\}$ no global max/min for $f(x,y) = x^3 - 3xy + y^3$

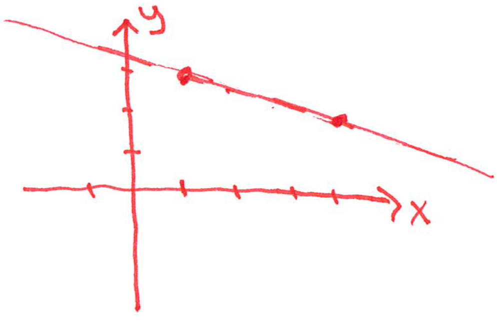
3 Constrained optimization

$$\max/\min \underbrace{f(x,y)}_{\text{objective function}} \quad \text{when} \quad \underbrace{g(x,y)=a}_{\text{constraint (equation)}}$$

Set D of admissible points $= \{(x,y) : g(x,y)=a\}$, set of points where the constraint is satisfied

Ex: $\max/\min f(x,y) = x^2+y^2$ when $x+3y=10$

D: $x+3y=10$



$\frac{3y}{3} = \frac{10-x}{3}$ $\quad y = \frac{10}{3} - \frac{1}{3}x$

Method of Lagrange multipliers:

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda(g(x,y)-a)$$
$$= x^2+y^2 - \lambda(x+3y-10)$$

$$\underbrace{\left.\begin{aligned} \mathcal{L}'_x &= 2x - \lambda\cdot 1 = 0 \\ \mathcal{L}'_y &= 2y - \lambda\cdot 3 = 0 \end{aligned}\right\} \quad \mathcal{L}'_\lambda = -1\cdot(x+3y-10) = 0 \}}_{\text{stationary pts of } \mathcal{L}}$$

$$\boxed{\begin{array}{ll} \text{FOC} & \mathcal{L}'_x = 0 \\ & \mathcal{L}'_y = 0 \\ \text{C} & x+3y=10 \end{array}}$$

Solutions of FOC+C (Lagrange conditions) $\rightsquigarrow$ candidate points

FOC: $L'_x = \boxed{2x - \lambda \cdot 1 = 0}$ $\quad 2x = \lambda \quad x = \lambda/2$

$L'_y = \boxed{2y - \lambda \cdot 3 = 0}$ $\quad 2y = 3\lambda \quad y = 3\lambda/2$

C: $\boxed{x + 3y = 10}$

$x + 3y = 10$

$\lambda/2 + 3(3\lambda/2) = 10 \quad | \cdot 2$

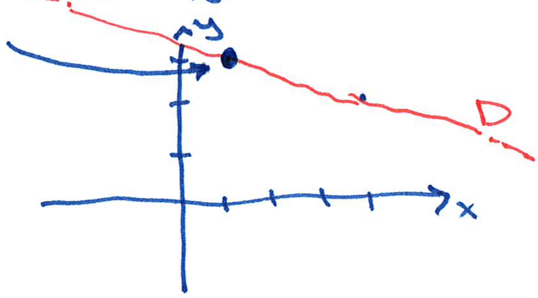
$x = 1 \quad y = 3 \quad \lambda = 2$

$\lambda + 9\lambda = 20$

$\frac{10\lambda}{10} = \frac{20}{10} \quad \lambda = 2$

Candidate pts: $(x, y; \lambda) = (1, 3; 2)$

$f = 10$



① If D is closed and bounded (compact set), then f has a max/min on D.

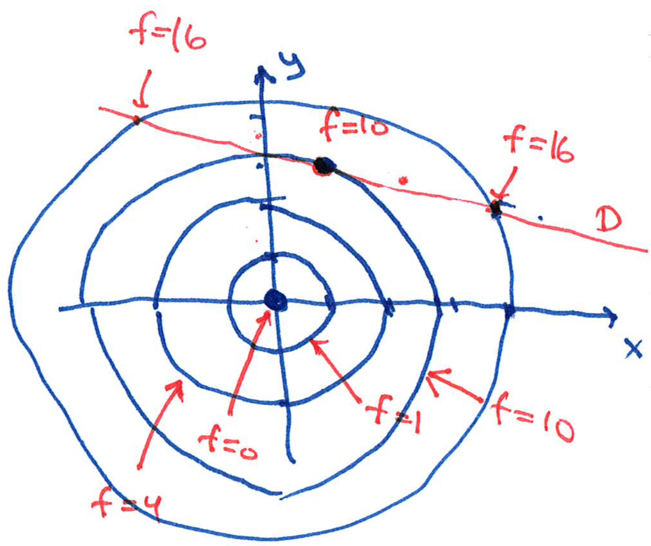
② Use level curves of f

$z = c$

$x^2 + y^2 = c$ $\quad$ $c = 0$: pt $(0,0)$

$c > 0$: circle, center $(0,0)$ radius $\sqrt{c}$

$c < 0$: no pts



Conclusion: $f_{min} = 10$ at $(1,3)$ with $\lambda = 2$

no max

What about:

min $f = x^2 + y^2$ when $x + 3y = 11$

new $f_{min} \simeq$ old $f_{min} + \lambda \cdot$ chage in a $= 10 + 2 \cdot 1 = 12$

Ex: max/min $f(x,y) = x+y$ when $x^3 - 3xy + y^3 = 0$

Lagrange:

$L = x+y - \lambda(x^3 - 3xy + y^3)$

$$\boxed{\begin{aligned} L'_x &= 1 - \lambda\cdot(3x^2 - 3y) = 0 \\ L'_y &= 1 - \lambda(-3x + 3y^2) = 0 \\ & x^3 - 3xy + y^3 = 0 \end{aligned}}$$

Cand. pt:

$(x,y;\lambda) = (3/2, 3/2; 4/9)$

$f=3$

(1) $1 = 3\lambda(x^2 - y)$

(2) $1 = 3\lambda(-x + y^2)$

(3) $x^3 - 3xy + y^3 = 0$

$3\lambda = \dfrac{1}{x^2-y}$

$3\lambda = \dfrac{1}{-x+y^2}$

$\Rightarrow \dfrac{1}{x^2-y} = \dfrac{1}{y^2-x}$

$y^2 - x = x^2 - y$

$y^2 - x^2 - x + y = 0$

$(y-x)(y+x) + (y-x) = 0$

$(y-x)(y+x+1) = 0$

$y = x$ or $x+y = -1 \to y = -1-x$

$x^3 - 3x^2 + x^3 = 0$

$2x^3 - 3x^2 = 0$

$x^2(2x-3) = 0$

$\cancel{x=0}$, $\cancel{y=0}$, $\cancel{\lambda=}$ or $x = 3/2$, $y = 3/2$, $\lambda = 4/9$

$\cancel{x^3 - 3x(-1-x) + (-1-x)^3 = 0}$

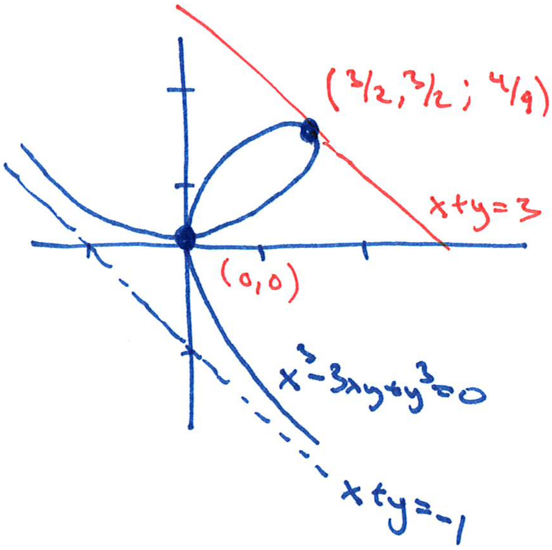
$\cancel{x^3 + 3x + 3x^2 + (-1) - 3x - 3x^2 - x^3 = 0}$

$\cancel{-1 = 0}$

$\lambda = \dfrac{1}{3}\cdot\dfrac{1}{x^2-y}$

$= \dfrac{1}{3}\cdot\dfrac{1\cdot 4}{\left(\frac{9}{4} - \frac{3}{2}\right)\cdot 4}$

$= \dfrac{4}{3\cdot 3} = \dfrac{4}{9}$



max/min $f(x,y)=x+y$ when $x^3-3xy+y^3=0$

$V_f=(-1,3]$ $f_{max}=3$
f_{min} : no min