Plan

1 Key Method: Determinants
2 Linear systems, inverse matrices and determinants

Review:

Any linear system ($m \times n$) can be written in matrix form as $A \cdot \underline{x} = \underline{b}$. It can also be represented by the augmented matrix $(A \mid \underline{b})$. The linear system can be solved used Gaussian elimination, and in general, it has either i) no solution ii) one unique solution, or iii) infinitely many solutions.

If A is invertible, then $A\underline{x} = \underline{b} \quad | A^{-1} \cdot \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b}$

In particular, the linear system has one unique solution if A is invertible.

$$I\underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b}$$

① Determinants

A ($n \times n$-matrix) $\rightsquigarrow$ $\det(A) = |A|$ determinant of A, a number.

Formulas:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} (-1)^{\operatorname{sgn}(\sigma)} a_{1\sigma(1)} \cdot \ldots \cdot a_{n\sigma(n)}$$

Ex.:
$$\begin{vmatrix} 1 & 2 \\ 7 & 4 \end{vmatrix} = 1 \cdot 4 - 7 \cdot 2 = 4 - 14 = \underline{-10}$$

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (A|\underline{b}) = \left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right)$$

2x2 lin. sys. coeff-matrix aug. matrix

$|A| = ad - bc$

Solving the linear system with Gaussian elim.:

$\underline{a \neq 0}$: $\left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right) \cdot a \to \left(\begin{array}{cc|c} a & b & e \\ ac & ad & af \end{array}\right) \big]{-c} \to \left(\begin{array}{cc|c} a & b & e \\ 0 & ad-bc & af-ce \end{array}\right)$

echelon form

$ad - bc \neq 0$: $\left(\begin{array}{cc|c} * & \cdot & \cdot \\ 0 & * & \cdot \end{array}\right)$ <u>one solution</u>

$ad - bc = 0$: $\left(\begin{array}{cc|c} * & \cdot & \cdot \\ 0 & 0 & * \end{array}\right)$ $af - ce \neq 0$: <u>no solutions</u>

$\left(\begin{array}{cc|c} * & \cdot & \cdot \\ 0 & 0 & 0 \end{array}\right)$ $af - ce = 0$: <u>infinitely many solutions</u>

<u>Result:</u> If $A\underline{x} = \underline{b}$ is a linear system (with A square), then:

$A\underline{x} = \underline{b}$ has one unique solution $\iff |A| \neq 0$

-"- has either no solutions or inf. many solutions $\iff |A| = 0$

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Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$

3x3 matrix

$|A| = \underline{1\cdot 2\cdot 9} + \underline{1\cdot 4\cdot 1} + \underline{1\cdot 1\cdot 3}$
$- \underline{1\cdot 2\cdot 1} - \underline{3\cdot 4\cdot 1} - \underline{9\cdot 1\cdot 1}$
$= 1\cdot(2\cdot 9 - 3\cdot 4) + 1\cdot(4\cdot 1 - 9\cdot 1)$
$+ 1\cdot(1\cdot 3 - 1\cdot 2)$

Methods for computing determinants:

(A) **Cofactor expansion:** along a row or column

Ex: $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1\cdot C_{11} + 1\cdot C_{12} + 1\cdot C_{13}$

cofactor expansion along the first row $= +1\cdot M_{11} - 1\cdot M_{12} + 1\cdot M_{13}$

sign $C_{ij} = (-1)^{i+j}\cdot M_{ij}$ $= +1\cdot\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

minor M_{ij} = determinant of the matrix you get when you delete row i, col. j

$= +(18-12) - (9-4) + (3-2) = 6 - 5 + 1 = \underline{\underline{2}}$

Note: - Cofactor expansion is a general method

- the cofactor expansion along any row or column gives the same result, det(A).

- in general, the expression for det(A) has $n!$ terms
$n! = n\cdot(n-1)(n-2)\cdots 2\cdot 1$

(B) Using Gaussian elimination

Ex:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = +1 \cdot \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 4 & 3 \\ 0 & 3 & 4 \\ -1 & 0 & 0 \end{vmatrix}$$

cofactor expansion

4x4 matrix

$$= +\left(+1 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}\right) + \left(-1 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}\right)$$

$$= \underline{\underline{0}}$$

Alt:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} +1 \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} -3/4$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

echelon form

$$4 - \frac{9}{4} = 7/4$$

Defn: A matrix is called upper triangular if all entries under the diagonal are zero. Note that all echelon forms are upper triangular.

Remark: The determinant of an upper triangular matrix is the product of the diagonal entries

$$|E| = 1 \cdot 4 \cdot 7/4 \cdot 0 = \underline{\underline{0}}$$

$$\parallel$$

$$|A| = \underline{\underline{0}}$$

Ex:

$$\begin{vmatrix} 17 & e^{\sqrt{\pi}} & -1634 \\ 0 & 3 & \sqrt{173} \\ 0 & 0 & 1 \end{vmatrix} = +17 \cdot \begin{vmatrix} 3 & \sqrt{173} \\ 0 & 1 \end{vmatrix}$$

$$= 17 \cdot (3 \cdot 1 - 0)$$

$$= 17 \cdot 3 \cdot 1 = \underline{\underline{51}}$$

Result: If $A \to B$ is an elementary row operation of the third type, then $|A| = |B|$.

If $A \to B$ is switching two rows, then $|B| = -|A|$

If $A \to B$ is multiplying one row with $c \neq 0$, then $|B| = c \cdot |A|$

Ex:

$$A = \begin{pmatrix} 1 & 7 & 3 & 5 \\ 2 & 5 & 1 & 4 \\ 3 & -4 & 0 & 1 \\ 4 & 3 & 3 & 6 \end{pmatrix} \begin{array}{l} \\ -2 \\ -3 \\ -4 \end{array} \to \begin{pmatrix} 1 & 7 & 3 & 5 \\ 0 & -9 & -5 & -6 \\ 0 & -25 & -9 & -14 \\ 0 & -25 & -9 & -14 \end{pmatrix} \begin{array}{l} \\ \\ \\ -1 \end{array} \to \begin{pmatrix} 1 & 7 & 3 & 5 \\ 0 & -9 & -5 & -6 \\ 0 & -25 & -9 & -14 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R(4) = R(1) + R(3)$

$|A| = \underline{\underline{0}}$

Note: The determinant of A is zero if either

i) A has a zero row

ii) A has two equal rows

iii) A has a row that we can obtained by adding multiples of other rows

The same applies to columns.

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(2) Inverse matrices and determinants

A $n \times n$-matrix

Result:

i) A is invertible $\Leftrightarrow |A| \neq 0 \Leftrightarrow$ Any linear system $A\underline{x} = \underline{b}$ has one unique sol.
(A^{-1} exists)

ii) If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

Transpose: A^T = transpose of A

Ex: $A = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \underline{v}^T = (1 \quad 2)$

Defn: A is called symmetric if $A^T = A$.

Ex: $A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 7 \end{pmatrix}$

Symmetric

$A^T = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 7 \end{pmatrix}$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

Alt I:

$$\underbrace{\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right)}_{= (A \mid I)} \to \cdots \to \underbrace{\left(\begin{array}{c|c} B & C \end{array}\right)}_{\text{reduced echelon form}}$$

Alt 2:

$$|A| = +1(18-12) - 1(9-3) + 1(4-2) = 6 - 6 + 2 = 2 \neq 0$$

$\to A^{-1}$ exists

$$C_{11} = +6 \quad C_{12} = -5 \quad C_{13} = +1$$
$$C_{21} = -6 \quad C_{22} = +8 \quad C_{23} = -2$$
$$C_{31} = +2 \quad C_{32} = -3 \quad C_{33} = +1$$

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)$$

$$= \frac{1}{2} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Key skills

- Do Gaussian elimination
- Compute matrix multiplication
- compute determinants

Remark: Some useful formulas

i) $|A \cdot B| = |A| \cdot |B|$

ii) $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

$(A \cdot B)^T = B^T A^T$

$$(B^{-1}A^{-1}) \cdot (AB) = B^{-1} A^{-1} A B$$
$$= B^{-1} I B = B^{-1} B = I$$
$$\cancel{(A^{-1}B^{-1})(AB) = A^{-1}B^{-1}AB}$$