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 Plan
 

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- 1 Key Method: Determinants
  - 2 Linear systems, inverse matrices and determinants
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Review:

Any linear system (max) can be written in matrix form as  $A \cdot \underline{x} = \underline{b}$ . It can also be represented by the augmented matrix  $(A|\underline{b})$ . The linear system can be solved using Gaussian elimination, and in general, it has either i) no solution, ii) one unique solution, or iii) infinitely many solutions.

If  $A$  is invertible, then  $A \underline{x} = \underline{b} \quad | \cdot A^{-1} \Rightarrow A^{-1} A \underline{x} = A^{-1} \underline{b}$   
 In particular, the linear system has one unique solution if  $A$  is invertible.  $\quad \underline{I} \underline{x} = A^{-1} \underline{b}$   
 $\quad \underline{x} = A^{-1} \underline{b}$

① Determinants

$A \rightsquigarrow \det(A) = |A|$   
 $n \times n$ -matrix  
 determinant of  $A$ ,  
 a number.

Formulas:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ = \underline{ad - bc}$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} a_{1\sigma(1)} \dots a_{n\sigma(n)}$$

Ex:  $\begin{vmatrix} 1 & 2 \\ 7 & 4 \end{vmatrix} = 1 \cdot 4 - 7 \cdot 2 \\ = 4 - 14 \\ = \underline{-10}$

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

2x2 lin. sys.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

coeff. matrix

$$|A| = ad - bc$$

$$(A|\underline{b}) = \left( \begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right)$$

aug. matrix

Solving the linear system with Gaussian elim.:

a ≠ 0:  $\left( \begin{array}{cc|c} \textcircled{a} & b & e \\ c & d & f \end{array} \right) \cdot a \rightarrow \left( \begin{array}{cc|c} \textcircled{a} & b & e \\ ac & ad & at \end{array} \right) \xrightarrow{-c} \left( \begin{array}{cc|c} \textcircled{a} & b & e \\ 0 & \textcircled{ad-bc} & at-ce \end{array} \right)$

echelon form

$ad - bc \neq 0$ :  $\left( \begin{array}{cc|c} \oplus & \oplus & \oplus \\ 0 & \oplus & \oplus \end{array} \right)$  one solution

$ad - bc = 0$ :  $\left( \begin{array}{cc|c} \oplus & \oplus & \oplus \\ 0 & 0 & \oplus \end{array} \right)$

$at - ce \neq 0$ : no solutions

$\left( \begin{array}{cc|c} \oplus & \oplus & \oplus \\ 0 & 0 & 0 \end{array} \right)$

$at - ce = 0$ : infinitely many solutions

Result: If  $A\underline{x} = \underline{b}$  is a linear system (with  $A$  square), then:

$A\underline{x} = \underline{b}$  has one unique solution  $\iff |A| \neq 0$

-||- has either no solutions or inf. many solutions.  $\iff |A| = 0$

Ex:  $A = \begin{pmatrix} 1 & 2 & 4 & 1 & 1 \\ 1 & 2 & 4 & 1 & 2 \\ 1 & 3 & 9 & 1 & 3 \end{pmatrix}$   
 3x3 matrix

$$|A| = \begin{aligned} & \underline{1 \cdot 2 \cdot 9} + \underline{1 \cdot 4 \cdot 1} + \underline{1 \cdot 1 \cdot 3} \\ & - \underline{1 \cdot 2 \cdot 1} - \underline{3 \cdot 4 \cdot 1} - \underline{9 \cdot 1 \cdot 1} \\ & = 1 \cdot (2 \cdot 9 - 3 \cdot 4) + 1 \cdot (4 \cdot 1 - 9 \cdot 1) \\ & \quad + 1 \cdot (1 \cdot 3 - 1 \cdot 2) \end{aligned}$$

Methods for computing determinants:

Ⓐ Cofactor expansion: along a row or column

Ex:  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$

cofactor expansion along the first row

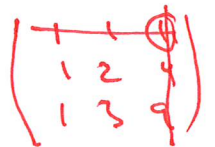
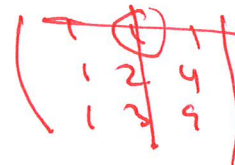
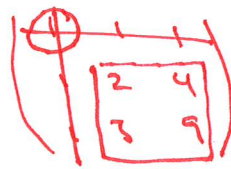
$$= +1 \cdot M_{11} - 1 \cdot M_{12} + 1 \cdot M_{13}$$

sign

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

$$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$



minor

$M_{ij}$  = determinant of the matrix you get when you delete row  $i$ , col.  $j$

$$= +(18-12) - (9-4) + (3-2) = 6 - 5 + 1 = \underline{\underline{2}}$$

Note: - Cofactor expansion is a general method

- the cofactor expansion along any row or column gives the same result,  $\det(A)$ .

- in general, the expression for  $\det(A)$  has  $n!$  terms

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$$

B Using Gaussian elimination

Ex:

cofactor expansion

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = +1 \cdot \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ -1 & 0 & 0 \end{vmatrix}$$

4x4 matrix

$$= + \left( +1 \cdot \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} \right) + \left( -1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \right)$$

$$= 0$$

Alt:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{+1} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-3/4}$$

Defn: A matrix is called upper triangular if all entries under the diagonal are zero. Note that all echelon forms are upper triangular.

$4 - \frac{3}{4} = \frac{7}{4}$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

echelon form

$$|E| = 1 \cdot 4 \cdot \frac{7}{4} \cdot 0 = 0$$

$$\Downarrow$$

$$|A| = 0$$

Remark: The determinant of an upper triangular matrix is the product of the diagonal entries

Ex:

$$\begin{vmatrix} 17 & e^{\sqrt{\pi}} & -1634 \\ 0 & 3 & \sqrt{173} \\ 0 & 0 & 1 \end{vmatrix} = +17 \cdot \begin{vmatrix} 3 & \sqrt{173} \\ 0 & 1 \end{vmatrix} \\ = 17 \cdot (3 \cdot 1 - 0) \\ = 17 \cdot 3 \cdot 1 = \underline{\underline{51}}$$

Result: If  $A \rightarrow B$  is an elementary row operation of the third type, then  $|A| = |B|$ .

If  $A \rightarrow B$  is switching two rows, then  $|B| = -|A|$

If  $A \rightarrow B$  is multiplying one row with  $c \neq 0$ ,  
then  $|B| = c \cdot |A|$

Ex:

$$A = \begin{pmatrix} 1 & 7 & 3 & 5 \\ 2 & 5 & 1 & 4 \\ 3 & -4 & 0 & 1 \\ 4 & 3 & 3 & 6 \end{pmatrix} \begin{matrix} \leftarrow -2 \\ \leftarrow -3 \\ \leftarrow -4 \end{matrix} \rightarrow \begin{pmatrix} 1 & 7 & 3 & 5 \\ 0 & -9 & -5 & -6 \\ 0 & -25 & -9 & -14 \\ 0 & -25 & -9 & -14 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow -1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 7 & 3 & 5 \\ 0 & -9 & -5 & -6 \\ 0 & -25 & -9 & -14 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R(4) = R(1) + R(3)$$

$$|A| = \underline{\underline{0}}$$

Note: The determinant of  $A$  is zero if either

- i)  $A$  has a zero row
- ii)  $A$  has two equal rows
- iii)  $A$  has a row that we can obtain by adding multiples of other rows

The same applies to columns.

## ② Inverse matrices and determinants

$A$   
 $n \times n$ -  
matrix

Result:

i)  $A$  is invertible  $\iff |A| \neq 0 \iff$  Any linear system  $A\underline{x} = \underline{b}$  has one unique sol.

ii) If  $|A| \neq 0$ , then

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$$

Transpose:  $A^T =$  transpose of  $A$

Ex:  $A = \begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$   $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 7 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \underline{v}^T = (1 \ 2)$$

Defn:

$A$  is called symmetric if  $A^T = A$ .

Ex:

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 7 \end{pmatrix}$$

Symmetric

$$A^T = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 7 \end{pmatrix}$$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

Alt 1:

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array} \right) \rightarrow \rightarrow \left( B \mid C \right)$$

reduced  
echelon  
form

Alt 2:

$$|A| = +1(1 \cdot 9 - 12) - 1(9 - 3) + 1(4 - 2) = 6 - 6 + 2 = 2 \neq 0$$

$\rightarrow A^{-1}$  exists

$$\begin{aligned} C_{11} &= +2 & C_{12} &= -5 & C_{13} &= +1 \\ C_{21} &= -2 & C_{22} &= +8 & C_{23} &= -2 \\ C_{31} &= +2 & C_{32} &= -3 & C_{33} &= +1 \end{aligned}$$

$$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$$

$$= \frac{1}{2} \cdot \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$$= \frac{1}{2} \begin{pmatrix} 6 & -5 & 1 \\ -2 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T = \frac{1}{2} \begin{pmatrix} 6 & -2 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Key skills

- Do Gaussian elimination
- Compute matrix multiplication
- compute determinants

Remark: Some useful formulas

i)  $|A \cdot B| = |A| \cdot |B|$

ii)  $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$

$(A \cdot B)^T = B^T A^T$

$$\begin{aligned} (B^T A^{-1}) \cdot (A B) &= B^T A^{-1} A B \\ &= B^T I B = B^T B = I \\ \cancel{(A^{-1} B^{-1}) \cdot (A B)} &= \cancel{A^{-1} B^{-1} A B} \end{aligned}$$